

**ON THE MORPHISMS AND TRANSFORMATIONS OF
TSUYOSHI FUJIWARA (AS A CONCRETION OF A BIDIMENSIONAL
MANY-SORTED GENERAL ALGEBRA AND ITS APPLICATION TO THE
EQUIVALENCE BETWEEN MANY-SORTED CLONES AND ALGEBRAIC
THEORIES)**

J. CLIMENT VIDAL AND J. SOLIVERES TUR

ABSTRACT. For, not necessarily similar, single-sorted algebras Fujiwara defined, through the concept of family of basic mapping-formulas between single-sorted signatures, a notion of morphism which generalizes the ordinary notion of homomorphism between algebras; and an equivalence relation, the conjugation, on the families of basic mapping-formulas, which corresponds to the relation of inner isomorphism for algebras. In this paper we extend the theory of Fujiwara about morphisms to the, not necessarily similar, many-sorted algebras, by defining the concept of polyderivator between many-sorted signatures, which assigns to basic sorts, words and to formal operations, families of derived terms, and under which the standard signature morphisms, the basic mapping-formulas of Fujiwara, and the derivators of Goguen-Thatcher-Wagner are subsumed. Then, by means of the homomorphisms between Bénabou algebras, which are the algebraic counterpart of the finitary many-sorted algebraic theories of Bénabou, we define the composition of polyderivators from which we get a corresponding category, and prove that it is isomorphic to the category of Kleisli for a monad on the standard category of many-sorted signatures. Next, by defining the notion of transformation between polyderivators, which generalizes the relation of conjugation of Fujiwara, we endow the category of many-sorted signatures and polyderivators with a structure of 2-category. From this we get a derived 2-category of many-sorted specifications in which we prove, syntactically, the equivalence of the many-sorted specifications of Hall and Bénabou, and, from a suitable pseudo-functor from it into the 2-category of categories, we deduce the equivalence of the categories of Hall and Bénabou algebras. Besides, by defining for each many-sorted signature its corresponding category of generalized many-sorted terms, we prove that the realization of these terms in the many-sorted algebras is invariant under polyderivators and compatible with the transformations between polyderivators, and from this we get an example, among others, of the new concept of 2-institution, itself an strict generalization of that of institution by Goguen and Burstall.

1. INTRODUCTION.

The closed sets of operations, or clones, on an arbitrary set A , i.e., the sets of operations on A closed under the generalized operations of composition and containing the projection mappings, were initially defined and investigated by P. Hall, as pointed out by Cohn in [14], pp. 127 and 132 (who attended the lectures by Professor P. Hall from 1944 to 1951), to show that the crucial mathematical

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properties of a Σ -algebra $\mathbf{A} = (A, (F_\sigma)_{\sigma \in \Sigma})$ do not depend on the family of primitive operations $(F_\sigma)_{\sigma \in \Sigma}$ on A defined by the single-sorted signature Σ , but on the system of all operations on A obtainable from $(F_\sigma)_{\sigma \in \Sigma}$ by means of the operations of composition.

The concept of an ordinary clone, axiomatized by P. Hall as a single-sorted partial algebra subject to satisfy some laws (see [14], p. 132) and, independently but subsequently, by M. Lazard as a compositor (see [41], p. 327), was generalized up to that of a many-sorted clone by Goguen and Meseguer in [26], and axiomatically defined by them (in [26], pp. 318–319) as any many-sorted algebra (of the appropriate signature) that satisfies a definite system of many-sorted equational laws, concretely, the so-called *Projection Axiom*, *Identity Axiom*, *Associativity Axiom*, and *Invariance of Constant Functions Axiom*. Given its origin in P. Hall, we agree to refer to the many-sorted algebras that are models of the just named axioms as Hall algebras.

Hall algebras, as reflected by the defining axioms, are a species of algebraic construct in which the essential properties of the fundamental procedures of *substitution*, for the many-sorted terms in the free many-sorted algebras, and of *composition*, for the many-sorted-operations on sorted sets are embodied. And this is precisely one of the reasons why Hall algebras are a powerful and fundamental instrument to investigate many-sorted algebras. To this we add that Hall algebras are not only worth of study because of its source in the above mentioned procedures. Besides that, Hall algebras are interesting in themselves since they furnish important examples of equationally defined many-sorted algebras, and also because they have been used by Goguen and Meseguer in [26] to prove the Completeness Theorem of finitary many-sorted equational logic (that generalizes the classical Completeness Theorem of finitary equational logic of Birkhoff), providing in this way, a full algebraization of many-sorted equational deduction.

Another approximation to the study of many-sorted algebras has been proposed by Bénabou in [2], by making use of the finitary many-sorted algebraic theories (categories with objects the words on a set of sorts S such that, for every word $w = (w_i)_{i \in n}$, there exists a family of morphisms $(p_i^w)_{i \in n}$, where, for $i \in n$, p_i^w is a morphism from w to (w_i) , the word of length one associated to the letter w_i , such that $(w, (p_i^w)_{i \in n})$ is a product of the family $((w_i)_{i \in n})$, that are the generalization to the many-sorted case of the finitary single-sorted algebraic theories of Lawvere, see [40]. The equational presentation of the finitary many-sorted algebraic theories of Bénabou gives rise to what we have called Bénabou algebras. And the Bénabou algebras, even having a many-sorted specification different from that of the Hall algebras, are also models of the essential properties of the clones for the many-sorted operations.

For an arbitrary, but fixed, set of sorts S , the many-sorted specifications H_S , for Hall algebras, and B_S , for Bénabou algebras, are *not isomorphic* in the category \mathbf{Spf} , of many-sorted specifications and many-sorted specification morphisms, because between the corresponding categories of models: $\mathbf{Alg}(H_S)$, of Hall algebras, and $\mathbf{Alg}(B_S)$, of Bénabou algebras, there is not any isomorphism. However, the many-sorted specifications H_S and B_S can be considered, in some definite way, as being *equivalent*, as a consequence of the proof, in the fourth section about Hall and Bénabou algebras, of the categorical equivalence between the categories $\mathbf{Alg}(H_S)$ and $\mathbf{Alg}(B_S)$.

But, the semantical equivalence of the many-sorted specifications H_S and B_S , or, for that matter, of any two many-sorted specifications, understood, by convention, as meaning the categorical equivalence of the canonically associated categories of models, can not be properly reflected at the purely syntactical level of

the many-sorted specifications and many-sorted specification morphisms, i.e., can not be mathematically defined in the category \mathbf{Spf} . And this is so, essentially, as a consequence of the fact of not having actually endowed \mathbf{Spf} with a (non trivial) structure of 2-category. Thus, if one remains anchored in the tradition of viewing \mathbf{Spf} as being, simply, a category, then the only reasonable way of classifying many-sorted specifications from within the category \mathbf{Spf} is through the category-theoretical concept of isomorphism, and not, by structural impossibility, by means of some other notion of equivalence between many-sorted specifications, itself being strictly weaker than that of isomorphism (as it would be the case if instead of having a category, we had a 2-category).

Therefore, what is really needed to settle the problem of the equivalence between many-sorted specifications (i.e., the problem of determining whether or not two many-sorted specifications determine equivalent categories) is to dispose of some way of comparing many-sorted specifications that goes, strictly, beyond the mere isomorphisms, in the same way as equivalences go beyond the isomorphisms when comparing categories among them. We suggest in this paper that an adequate way of providing a solution to the just mentioned problem is by constructing suitable 2-categories of many-sorted signatures and many-sorted specifications, through the appropriate definitions of the 2-cells between the 1-cells. This bidimensionality, by supplying one additional degree of freedom, generates a richer world, that opens the possibility to deal not only with isomorphic but also with adjoint and equivalent many-sorted specifications. Thus carrying further the previous development which was incomplete because of its restriction to categories. The methodology we have followed in order to find a solution of the equivalence problem will now be considered.

It consists in generalizing the theory of Fujiwara in [21] and [22] into several directions. Firstly, by defining the concept of *morphism of Fujiwara*, from now on abbreviated to *polyderivator*, from a many-sorted signature into another, which assigns to basic sorts, words and to formal operations, families of derived terms, and this in such a way that under the concept of polyderivator falls the concept of derivator, defined in [27], and that of morphism between many-sorted algebraic theories. Secondly, by endowing with a structure of 2-category the category of many-sorted signatures and polyderivators, by defining the appropriate transformations between the polyderivators, that generalize the equivalences defined by Fujiwara in [22], and allow richer comparisons between many-sorted signatures than the usually considered. Lastly, by introducing the corresponding 2-categories of many-sorted specifications, polyderivators between many-sorted specifications and transformations from such a polyderivator into a like one.

After having developed the generalized theory we prove that the transformations between polyderivators determine natural transformations between the functors, on the categories of many-sorted algebras and many-sorted terms, associated to the polyderivators. Besides, we prove that the realization of the many-sorted terms in the many-sorted algebras is invariant under the polyderivators and compatible with the transformations between the polyderivators, from which we get an example of 2-institution, that generalizes the usual notion of institution as defined in [25].

By using the machinery introduced we prove, as an example, the equivalence between the many-sorted specifications of Hall and Bénabou. And from this we get, as an immediate consequence of the existence of a certain pseudo-functor from the 2-category \mathbf{Spf}_{ps} , of many-sorted specifications, to the 2-category \mathbf{Cat} , the equivalence between the categories of Hall and Bénabou algebras. This, we believe, helps to understand, from a purely category-theoretical standpoint, how some equivalences between categories, e.g., that between clones (represented by Hall algebras)

and finitary many-sorted algebraic theories (represented by Bénabou algebras), arise from more primitive syntactical equivalences between some many-sorted specifications associated to them.

Note that the suggested solution to the equivalence problem between many-sorted specifications bears some resemblance in *intention* to e.g., the classification of the homology theories (functors from a topological category to an algebraic category satisfying some axioms) accomplished by Eilenberg and Steenrod in [17], an impossible task without the notion of natural equivalence (invertible 2-cell) of functors (1-cells).

The paper falls naturally into two parts. The first part is divided into three sections: INTRODUCTION, common to both parts, THE MANY-SORTED TERM INSTITUTION, and MANY-SORTED SPECIFICATIONS AND MORPHISMS; and the second is divided into four sections: HALL AND BÉNABOU ALGEBRAS, MORPHISMS OF FUJIWARA, TRANSFORMATIONS OF FUJIWARA, and EQUIVALENCE OF THE SPECIFICATIONS OF HALL AND BÉNABOU. Next we proceed to describe the contents of the just enumerated sections, leaving out, obviously, the first one.

The main goal of the *second section* is to construct the many-sorted term institution. To attain such a goal we begin by defining \mathbf{MSet} , the category of many-sorted sets and many-sorted mappings, \mathbf{Sig} , the category of standard many-sorted signatures, and \mathbf{Alg} , the category of standard many-sorted algebras, through the construction of Ehresmann-Grothendieck applied, respectively, to the contravariant functors \mathbf{MSet} , \mathbf{Sig} , and \mathbf{Alg} . Then we remark that \mathbf{MSet} and \mathbf{Sig} are split bifibrations on \mathbf{Set} and prove that \mathbf{MSet} , \mathbf{Sig} , and \mathbf{Alg} are bicomplete, that \mathbf{Alg} is concrete, univocally transportable through a “forgetful” functor G into the fibered product $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, and that the functor G has a left adjoint $\mathbf{T}: \mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig} \longrightarrow \mathbf{Alg}$ which transforms objects of $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ into labelled term algebras in \mathbf{Alg} and morphisms of $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ into translators between the associated labelled term algebras in \mathbf{Alg} .

On the basis of the functor \mathbf{T} we define, for every many-sorted signature Σ , the category $\mathbf{Ter}(\Sigma)$, of generalized terms for Σ , as the dual of the Kleisli category for \mathbb{T}_{Σ} (the standard monad derived from the adjunction between the category $\mathbf{Alg}(\Sigma)$ and the category \mathbf{Set}^S), and we extend this procedure up to a pseudo-functor \mathbf{Ter} from \mathbf{Sig} to \mathbf{Cat} which formalizes the procedure of translation for many-sorted terms. Then, to account exactly for the invariant character of the process of realization of the many-sorted terms in the many-sorted algebras, under change of many-sorted signature, we show that there exists a pseudo-extranatural transformation from a pseudo-functor obtained from \mathbf{Alg} and \mathbf{Ter} to the constant functor $K_{\mathbf{Set}}$, both defined on $\mathbf{Sig}^{\text{op}} \times \mathbf{Sig}$ and taking values in the 2-category \mathbf{Cat} . Finally, after generalizing the concept of institution by means, essentially, of the notion of pseudo-extranatural transformation from a pseudo-functor to a constant functor, we get \mathfrak{Tm} , the many-sorted term institution on \mathbf{Set} .

In the *third section* we begin by defining, for a many-sorted signature Σ , the concept of Σ -equation, but for the generalized terms in $\mathbf{Ter}(\Sigma)$, the relation of satisfaction between many-sorted algebras and Σ -equations, the consequence operator \mathbf{Cn}_{Σ} , and by translating, for a morphism between many-sorted signatures, equations for the source many-sorted signature into equations for the target many-sorted signature. Then we continue with the proof of the satisfaction condition and, after defining a convenient pseudo-functor from \mathbf{Sig} to $\mathbf{Cat}_{\mathcal{V}}$, for an adequate Grothendieck universe \mathcal{V} , we get the equational institution on $\mathbf{2}$.

Next, in order to explain category-theoretically the concept of equational deduction, we begin by defining, by means of the construction of Ehresmann-Grothendieck applied to a suitable contravariant functor from \mathbf{Set} to \mathbf{Cat} , the category \mathbf{MCISp} ,

of many-sorted closure spaces. Then, through the concept of adjoint square, we define, for the Grothendieck universe \mathcal{V} , the 2-category $\mathbf{Mnd}_{\mathcal{V},\text{alg}}$ of monads, algebraic morphisms between monads, and transformations between algebraic morphisms, into which the category \mathbf{MCISp} is naturally embedded. After this we prove the existence of a pseudo-functor \mathbf{Cn} from \mathbf{Sig} to $\mathbf{Mnd}_{\mathcal{V},\text{alg}}$ that has as components, essentially, the consequence operators \mathbf{Cn}_{Σ} for the different signatures Σ , and make it, after generalizing the concept of entailment system, part of the, so-called, equational consequence entailment system.

Following this, after defining the category \mathbf{Spf} , of many-sorted specifications and many-sorted specification morphisms, we prove the existence of a contravariant functor, \mathbf{Alg}^{sp} and of a pseudo-functor, \mathbf{Ter}^{sp} , from \mathbf{Spf} to \mathbf{Cat} , that extend \mathbf{Alg} and \mathbf{Ter} , respectively. Then we state that from $\mathbf{Spf}^{\text{op}} \times \mathbf{Spf}$ to the 2-category \mathbf{Cat} there exists a pseudo-functor, obtained from \mathbf{Alg}^{sp} and \mathbf{Ter}^{sp} , and a pseudo-extranatural transformation from it to the functor constantly \mathbf{Set} , and from this we get \mathfrak{Spf} , the many-sorted specification institution of Fujiwara on \mathbf{Set} , and an institution morphism from \mathfrak{Spf} to \mathfrak{Im} .

In the *fourth section* we show that the categories of Hall and Bénabou algebras, that are models of the essential properties of the clones of many-sorted operations, are equivalent and also that there exists a biunivocal correspondence between the Bénabou algebras and the finitary many-sorted Bénabou theories, that are a generalization of the single-sorted algebraic theories of Lawvere. Furthermore, these algebras are interesting because they will be used, among other things, to define, later on, the composition of the polyderivors from a many-sorted signature into another, that are generalizations of the usual morphisms between many-sorted signatures, and also to exemplify an equivalence between the specifications of Hall and Bénabou algebras in a 2-category of specifications, polyderivors, and transformations.

In the *fifth section* after defining the morphisms of Fujiwara between many-sorted signatures (that generalize the standard morphisms and the derivors between many-sorted signatures, as well as the families of basic mapping-formulas defined by Fujiwara in [21] for the single-sorted case), and the composition of these morphisms, we get the category \mathbf{Sig}_{pd} , of many-sorted signatures and morphisms of Fujiwara, and prove that it can be obtained, essentially, as the Kleisli category for a monad in \mathbf{Sig} . Then we define a pseudo-functor (contravariant in the morphisms) $\mathbf{Alg}_{\text{pd}} : \mathbf{Sig}_{\text{pd}} \longrightarrow \mathbf{Cat}$ and, by applying the construction of Ehresmann-Grothendieck, we get a new category \mathbf{Alg}_{pd} of many-sorted algebras and morphisms between many-sorted algebras which have, as a component, the morphisms of Fujiwara. Following this we define another pseudo-functor (covariant in the morphisms) \mathbf{Ter}_{pd} from \mathbf{Sig}_{pd} to \mathbf{Cat} which formalizes the procedure of translation for many-sorted terms, but now for the morphisms of Fujiwara. Then, to account exactly for the invariant character of the process of realization of the many-sorted terms in the many-sorted algebras, under change of many-sorted signature through the morphisms of Fujiwara, we show that there exists a pseudo-extranatural transformation from a pseudo-functor obtained from \mathbf{Alg}_{pd} and \mathbf{Ter}_{pd} to the constant functor $\mathbf{K}_{\mathbf{Set}}$, both defined on $\mathbf{Sig}_{\text{pd}}^{\text{op}} \times \mathbf{Sig}_{\text{pd}}$ and taking values in the 2-category \mathbf{Cat} , and from this we get $\mathfrak{Im}_{\text{pd}}$, the many-sorted term institution of Fujiwara.

In the *sixth section* we endow the category \mathbf{Sig}_{pd} with a structure of 2-category through the concept of transformation between morphisms of Fujiwara, which generalizes that one of equivalence between families of basic mapping-formulas, defined by Fujiwara in [22] for the single-sorted case. Then we prove that the transformations between morphisms of Fujiwara determine natural transformations between the functors associated to the morphisms of Fujiwara. From this we extend the

pseudo-functors $\text{Alg}_{\mathfrak{p}\mathfrak{d}}$ and $\text{Ter}_{\mathfrak{p}\mathfrak{d}}$ up to the 2-category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$, and we get, in particular, by applying a construction of Ehresmann-Grothendieck to $\text{Alg}_{\mathfrak{p}\mathfrak{d}}$, a corresponding 2-category $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$. Next, after proving that the transformations between morphisms of Fujiwara are compatible with the realization of the many-sorted terms in the many-sorted algebras, we show that there exists a pseudo-extranatural transformation from a pseudo-functor obtained from $\text{Alg}_{\mathfrak{p}\mathfrak{d}}$ and $\text{Ter}_{\mathfrak{p}\mathfrak{d}}$ to the constant functor $\mathbf{K}_{\mathbf{Set}}$, both defined on the 2-category $\mathbf{Sig}^{\text{op}} \times \mathbf{Sig}$ and taking values in the 2-category \mathbf{Cat} , and from this we get $\mathfrak{Tm}_{\mathfrak{p}\mathfrak{d}}$, the many-sorted term 2-institution of Fujiwara.

Besides, we prove that the morphisms and transformations of Fujiwara can be taken as a concretion of a bidimensional many-sorted general algebra, on the basis of the existence of an embedding from the 2-category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ into the 2-category $\mathbf{Mnd}_{\mathbf{V}, \text{alg}}$ which sends a signature Σ to the monad $(\mathbf{Set}^S, T_\Sigma)$, a polyderivator \mathbf{d} from Σ to Λ to the associate algebraic morphism $T_{\mathbf{d}}$ from $(\mathbf{Set}^S, T_\Sigma)$ to $(\mathbf{Set}^T, T_\Lambda)$, and a transformation ξ from \mathbf{d} to \mathbf{e} to the corresponding algebraic transformation T_ξ from $T_{\mathbf{d}}$ to $T_{\mathbf{e}}$.

In the *seventh section* we define a 2-category of specifications, $\mathbf{Spf}_{\mathfrak{p}\mathfrak{d}}$, with objects the specifications, 1-cells from a specification into a like one the polyderivators between the underlying signatures of the specifications that are compatible with the equations, and 2-cells from a 1-cell into a like one a convenient class of transformations between the polyderivators. Following this we prove that the contravariant pseudo-functor $\text{Alg}_{\mathfrak{p}\mathfrak{d}}$ and the pseudo-functor $\text{Ter}_{\mathfrak{p}\mathfrak{d}}$, both defined on the 2-category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$, can be lifted up to the 2-category $\mathbf{Spf}_{\mathfrak{p}\mathfrak{d}}$ as $\text{Alg}_{\mathfrak{p}\mathfrak{d}}^{\text{sp}}$ and $\text{Ter}_{\mathfrak{p}\mathfrak{d}}^{\text{sp}}$, respectively. Then we state that from the 2-category $\mathbf{Spf}_{\mathfrak{p}\mathfrak{d}}^{\text{op}} \times \mathbf{Spf}_{\mathfrak{p}\mathfrak{d}}$ to the 2-category \mathbf{Cat} there exists a pseudo-functor, obtained from $\text{Alg}_{\mathfrak{p}\mathfrak{d}}^{\text{sp}}$ and $\text{Ter}_{\mathfrak{p}\mathfrak{d}}^{\text{sp}}$, and a pseudo-extranatural transformation from it to the functor constantly \mathbf{Set} , and from this we get $\mathfrak{Spf}_{\mathfrak{p}\mathfrak{d}}$, the many-sorted specification 2-institution of Fujiwara.

Finally, it is in the 2-category $\mathbf{Spf}_{\mathfrak{p}\mathfrak{d}}$ that we prove, for every set of sorts S , the equivalence of the specifications of Hall and Bénabou for S , from which, through the pseudo-functor $\text{Alg}_{\mathfrak{p}\mathfrak{d}}^{\text{sp}}$, the equivalence between the corresponding categories of algebras, $\mathbf{Alg}(\mathbf{H}_S)$ and $\mathbf{Alg}(\mathbf{B}_S)$, follows immediately.

Every set we consider, unless otherwise stated, will be a \mathbf{U} -small set or a \mathbf{U} -large set, i.e., an element or a subset, respectively, of a Grothendieck(-Sonner-Tarski) universe \mathbf{U} (as defined in [42], p. 22, but see also [49], pp. 160–163, [55], pp. 166–167, and [60], p. 84, for more details about this concept), fixed once and for all. Besides, we agree that \mathbf{Set} denotes the category which has as set of objects \mathbf{U} and as set of morphisms the subset of \mathbf{U} of all mappings between \mathbf{U} -small sets, and, depending on the context, that \mathbf{Cat} denotes either, the category of the \mathbf{U} -categories (i.e., categories \mathbf{C} such that the set of objects of \mathbf{C} is a subset of the Grothendieck universe \mathbf{U} , and the hom-sets of \mathbf{C} elements of \mathbf{U}), and functors between \mathbf{U} -categories, or the 2-category of the \mathbf{U} -categories, functors between \mathbf{U} -categories, and natural transformations between functors.

In all that follows we use standard concepts and constructions from category theory, see e.g., [5], [6], [7], [16], [30], [31], [38], [42], and [50]; classical universal algebra, see e.g., [14], [28], [37], [39], and [46]; categorical universal algebra, see e.g., [2] and [40]; many-sorted algebra, see e.g., [2], [4], [26], [32] (particularly Chapter 3), [34], [43], [45] and [48]; and set theory, see e.g., [8], [18] and [49]. Nevertheless, we have generically adopted the following notational and conceptual conventions:

- (1) As far as a category \mathbf{C} is concerned we will write, for an object x of \mathbf{C} , $x \in \mathbf{C}$ instead of $x \in \text{Ob}(\mathbf{C})$, however, for a morphism f of \mathbf{C} , we will write $f \in \text{Mor}(\mathbf{C})$ but not $f \in \mathbf{C}$. Furthermore, if there is no risk of confusion, in order to simplify the notation, we will write 1 for the identity functor at \mathbf{C} ,

F for the identity natural transformation at the functor F , and, under the same circumstance and reason, we will use the juxtaposition to denote both the horizontal and the vertical composition of natural transformations.

- (2) Relative to set theory we recall that between ordinals $<$ is identified with \in ; thus for a (*von Neumann*) ordinal α we have that $\alpha = \{\beta \mid \beta \in \alpha\}$, and \mathbb{N} , the first transfinite ordinal, is the set of all *natural numbers*. For two sets A, B we denote by $\text{Hom}(A, B)$ the set of all *mappings* $f: A \longrightarrow B$ from A to B , i.e., the set of all ordered triples $f = (A, F, B)$ where F is a *function* from A to B . For a mapping $f: A \longrightarrow B$, a subset X of A , and a subset Y of B , we denote by $f^{-1}[Y]$ the *inverse image of Y under f* , and by $f[X]$ the *direct image of X under f* . However, if $Y = \{y\}$ is a *final set*, we will write $f^{-1}[y]$ instead of the more accurate $f^{-1}[\{y\}]$. For sets B, C , a family of sets $(A_i)_{i \in I}$, a family of mappings $(f_i)_{i \in I}$ in $\prod_{i \in I} \text{Hom}(B, A_i)$, and a family of mappings $(g_i)_{i \in I}$ in $\prod_{i \in I} \text{Hom}(A_i, C)$, we denote by $\langle f_i \rangle_{i \in I}$, resp., by $[g_i]_{i \in I}$, the unique mapping from B to $\prod_{i \in I} A_i$, resp., from $\prod_{i \in I} A_i$ to C , such that, for every $i \in I$, $f_i = \text{pr}_i \circ \langle f_i \rangle_{i \in I}$, resp., $g_i = [g_i]_{i \in I} \circ \text{in}_i$.
- (3) For a set S we agree upon denoting by $\mathbf{T}_*(S) = (S^*, \lambda, \lambda)$ the *free monoid on S* , where

- (a) S^* , the underlying set of $\mathbf{T}_*(S)$, is $\bigcup_{n \in \mathbb{N}} S^n$, the set of all *words on S* , with S^n the set of all *functions* from n to S ,
- (b) λ , the *concatenation* of words on S , is the binary operation on S^* which sends a pair of words (w, v) on S to the function $w \lambda v$ from $|w| + |v|$ to S , where $|w|$ and $|v|$ are the lengths of w and v , respectively, defined as follows

$$w \lambda v \begin{cases} |w| + |v| \longrightarrow S \\ i \longmapsto \begin{cases} w_i, & \text{if } 0 \leq i < |w|; \\ v_{i-|w|}, & \text{if } |w| \leq i < |w| + |v|, \end{cases} \end{cases}$$

and

- (c) λ is the *empty word*, i.e., the unique function from 0 to S .

For a mapping $\varphi: S \longrightarrow T$ we denote by φ^* the unique homomorphism from $\mathbf{T}_*(S)$ to $\mathbf{T}_*(T)$ such that $\varphi^* \circ \check{\imath}_S = \check{\imath}_T \circ \varphi$, where $\check{\imath}_S$, resp., $\check{\imath}_T$, is the canonical embedding of S into S^* , resp., of T into T^* .

More specific notational conventions will be included and explained in the successive sections.

We point out that in, almost, all that follows we frequently draw diagrams to provide a geometrical description of what is going on. By doing so we hopefully expect to aid the reader in his understanding of the notions and constructions involved in each concrete situation, as well as in his grasping of the displayed ideas (after all, diagrams are intended precisely for that purpose). Finally, we warn the reader that dealing with entities depending on a great number of parameters which, in addition, are allowed to vary simultaneously, has indeed a high notational price (to say nothing of the fact that living and working in the topos \mathbf{MSet} —which, in particular, is non-Boolean, has as internal logic the trivalued logic of Heyting (see, e.g., [33]), and is of De Morgan—and associated categories is much more demanding than it is in \mathbf{Set}^S —which, among others properties, is a Boolean topos—and associated categories, for an arbitrary, but *fixed*, set of sorts S). However, we have tried our best to do the notation as uniform, simple, and clear as possible.

2. THE MANY-SORTED TERM INSTITUTION.

Our main aim in this section is to show that the concept of “derived operation of an algebra”, also known as “term operation of an algebra”, elemental as it is,

but fundamental for universal algebra, can be naturally subsumed under the notion of institution (see for this notion, e.g., [25]), provided that an institution is meant not to be an extranatural transformation (as in [25]) but a pseudo-extranatural transformation (as defined at the end of this section).

To attain the aim just mentioned we begin by a careful examination of the different types of things that are involved around it, namely many-sorted sets, signatures, algebras, terms, and generalized institutions. More specifically, in this section we define the category **MSet** of many-sorted sets, in which the many-sorted sets will be labelled with the sets of sorts, by applying the construction of (at least) Ehresmann-Grothendieck (see [16], pp. 89–91, where it is called “produit croisé”, and [31], pp. (sub.) 175–177) to a contravariant functor from **Set** to **Cat**. Following this we define the categories **Sig**, of many-sorted signatures, and **Alg**, of many-sorted algebras, by applying also the construction of Ehresmann-Grothendieck to suitable contravariant functors from **Set** and **Sig**, respectively, to **Cat**.

Besides we prove the existence of a left adjoint **T** to a “forgetful” functor **G** from **Alg** to **MSet** $\times_{\mathbf{Set}}$ **Sig**, and from this left adjoint **T** we define a pseudo-functor **Ter** from **Sig** to **Cat** which formalizes the procedure of translation for many-sorted terms.

Finally, to account exactly for the invariant character of the realization of many-sorted terms in many-sorted algebras under change of many-sorted signature, we prove the existence of a pseudo-extranatural transformation (Tr, θ) from a pseudo-functor $\text{Alg}(\cdot) \times \text{Ter}(\cdot)$, from **Sig**^{op} \times **Sig** to **Cat**, to the functor $K_{\mathbf{Set}}$, between the same categories, that is constantly **Set**. Then, after providing a generalization of the ordinary concept of institution, we prove that (Tr, θ) is part of an institution on **Set**, the so-called many-sorted term institution.

Before stating the first proposition of this section, we agree upon calling, from now on, for a set (of sorts) $S \in \mathcal{U}$, the objects of the category **Set**^{*S*} (i.e., the functions $A = (A_s)_{s \in S}$ from S to \mathcal{U}) *S*-sorted sets; and the morphisms of the category **Set**^{*S*} from an *S*-sorted set A into a like one B (i.e., the ordered triples (A, f, B) , abbreviated to $f: A \longrightarrow B$, where f is an element of $\prod_{s \in S} \text{Hom}(A_s, B_s)$, the cartesian product of the family $(\text{Hom}(A_s, B_s))_{s \in S}$) *S*-sorted mappings from A to B . Furthermore, we also agree that a pseudo-functor F from a category **C** to a 2-category **D** consists of the following data:

- (1) An object mapping $F: \text{Ob}(\mathbf{C}) \longrightarrow \text{Ob}(\mathbf{D})$.
- (2) For every $x, y \in \mathbf{C}$, an hom-mapping

$$F: \text{Hom}_{\mathbf{C}}(x, y) \longrightarrow \text{Hom}_{\mathbf{D}}(F(x), F(y)).$$

- (3) For every morphisms $f: x \longrightarrow y$ and $g: y \longrightarrow z$ in **C**, an isomorphic 2-cell $\gamma^{f,g}$ from $F(g) \circ F(f)$ to $F(g \circ f)$.
- (4) For every $x \in \mathbf{C}$, an isomorphic 2-cell ν^x from $\text{id}_{F(x)}$ to $F(\text{id}_x)$.

These data must satisfy the following coherence axioms:

- (1) For morphisms $f: x \longrightarrow y$, $g: y \longrightarrow z$, and $h: z \longrightarrow t$ in **C**,

$$\gamma^{g \circ f, h} \circ (\text{id}_{F(h)} * \gamma^{f, g}) = \gamma^{f, h \circ g} \circ (\gamma^{g, h} * \text{id}_{F(f)}).$$

- (2) For a morphism $f: x \longrightarrow y$ in **C**,

$$\text{id}_{F(f)} = \gamma^{\text{id}_x, f} \circ (\text{id}_{F(f)} * \nu^x) \quad \text{and} \quad \text{id}_{F(f)} = \gamma^{f, \text{id}_y} \circ (\nu^y * \text{id}_{F(f)}).$$

In the following proposition, that is basic for a great deal of what follows, for a mapping φ from S to T , we prove the existence of an adjunction $\prod_{\varphi} \dashv \Delta_{\varphi}$ from the category of *S*-sorted sets to the category of *T*-sorted sets, as well as the existence of a contravariant functor **MSet** and of a pseudo-functor **MSet**^{**I**} (related, respectively, to the right and left components of the adjunction) from **Set** to **Cat**.

Proposition 1. *Let $\varphi: S \longrightarrow T$ be a mapping. Then the functors Δ_φ from \mathbf{Set}^T to \mathbf{Set}^S and \coprod_φ from \mathbf{Set}^S to \mathbf{Set}^T defined, respectively, as follows*

- (1) Δ_φ assigns to a T -sorted set A the S -sorted set $A_\varphi = (A_{\varphi(s)})_{s \in S}$, i.e., $A \circ \varphi$, and to a T -sorted mapping $f: A \longrightarrow B$ the S -sorted mapping

$$f_\varphi = (f_{\varphi(s)})_{s \in S}: A_\varphi \longrightarrow B_\varphi;$$

- (2) \coprod_φ assigns to an S -sorted set A the T -sorted set $\coprod_\varphi A = (\coprod_{s \in \varphi^{-1}[t]} A_s)_{t \in T}$ and to an S -sorted mapping $f: A \longrightarrow B$ the T -sorted mapping

$$\coprod_\varphi f = (\coprod_{s \in \varphi^{-1}[t]} f_s)_{t \in T}: \coprod_\varphi A \longrightarrow \coprod_\varphi B,$$

are such that, $\coprod_\varphi \dashv \Delta_\varphi$. We agree that θ^φ , η^φ , and ε^φ denote, respectively, the natural isomorphism, the unit and the counit of the adjunction.

Besides, there exists a contravariant functor \mathbf{MSet} from \mathbf{Set} to \mathbf{Cat} which sends a set S to the category $\mathbf{MSet}(S) = \mathbf{Set}^S$, and a mapping φ from S to T to the functor Δ_φ from \mathbf{Set}^T to \mathbf{Set}^S ; and a pseudo-functor $\mathbf{MSet}^{\mathbb{I}}$ from \mathbf{Set} to the 2-category \mathbf{Cat} given by the following data

- (1) The object mapping of $\mathbf{MSet}^{\mathbb{I}}$ is that which sends a set S to the category $\mathbf{MSet}^{\mathbb{I}}(S) = \mathbf{Set}^S$.
- (2) The morphism mapping of $\mathbf{MSet}^{\mathbb{I}}$ is that which sends a mapping φ from S to T to the functor $\mathbf{MSet}^{\mathbb{I}}(\varphi) = \coprod_\varphi$ from \mathbf{Set}^S to \mathbf{Set}^T .
- (3) For every $\varphi: S \longrightarrow T$ and $\psi: T \longrightarrow U$, the natural isomorphism $\gamma^{\varphi, \psi}$ from $\coprod_\psi \circ \coprod_\varphi$ to $\coprod_{\psi \circ \varphi}$ is that which is defined, for every S -sorted set A , as the U -sorted mapping that in the u -th coordinate is $((a, s), \varphi(s)) \mapsto (a, s)$, if there exists an $s \in S$ such that $u = \psi(\varphi(s))$, and is the identity at \emptyset , otherwise.
- (4) For every set S , the natural isomorphism ν^S from $\mathbf{Id}_{\mathbf{Set}^S}$ to \coprod_{id_S} is that which is defined, for every S -sorted set A and $s \in S$, as the canonical isomorphism from A_s to $A_s \times \{s\}$.

Proof. We begin by proving that \coprod_φ is a left adjoint to Δ_φ . Let A be an S -sorted set, then the pair $(\eta_A^\varphi, \coprod_\varphi A)$, where η_A^φ is the S -sorted mapping from A to $\Delta_\varphi(\coprod_\varphi A) = (\coprod_{x \in \varphi^{-1}[\varphi(s)]} A_x)_{s \in S}$ whose s -th coordinate, for $s \in S$, is the canonical embedding from A_s to $\coprod_{x \in \varphi^{-1}[\varphi(s)]} A_x$, is a universal morphism from A to Δ_φ . This is so because, for a T -sorted set B and an S -sorted mapping $f: A \longrightarrow B_\varphi$, the T -sorted mapping $f^\S = (f_t^\S)_{t \in T}$ from $\coprod_\varphi A$ to B , where, for $t \in T$, f_t^\S is the unique mapping $[f_s]_{s \in \varphi^{-1}[t]}$ from $\coprod_{s \in \varphi^{-1}[t]} A_s$ to $B_t = B_{\varphi(s)}$ such that, for every $s \in \varphi^{-1}[t]$, the following diagram commutes

$$\begin{array}{ccc} A_s & \xrightarrow{\text{in}_s} & \coprod_{s \in \varphi^{-1}[t]} A_s \\ & \searrow f_s & \downarrow [f_s]_{s \in \varphi^{-1}[t]} = f_t^\S \\ & & B_t = B_{\varphi(s)} \end{array}$$

is such that $f = \Delta_\varphi(f^\S) \circ \eta_A^\varphi$ and unique with such a property.

Otherwise stated, $\coprod_\varphi A$ is, simply, $\text{Lan}_\varphi A$, i.e., the left Kan extension of A along φ , recalling that every set is the set of objects of a discrete category and every mapping between sets the object mapping of a functor between discrete categories.

To prove that $\mathbf{MSet}^{\mathbb{I}}$ is a pseudo-functor, it is enough to verify the coherence axioms. But given the situation

$$S \xrightarrow{\varphi} T \xrightarrow{\psi} U \xrightarrow{\xi} X,$$

the following diagrams commute

$$\begin{array}{ccc} \coprod_{\xi} \circ \coprod_{\psi} \circ \coprod_{\varphi} & \xrightarrow{\text{id}_{\coprod_{\xi}} * \gamma^{\varphi, \psi}} & \coprod_{\xi} \circ \coprod_{\psi \circ \varphi} \\ \downarrow \gamma^{\psi, \xi} * \text{id}_{\coprod_{\varphi}} & & \downarrow \gamma^{\psi \circ \varphi, \xi} \\ \coprod_{\xi \circ \psi} \circ \coprod_{\varphi} & \xrightarrow{\gamma^{\varphi, \xi \circ \psi}} & \coprod_{\xi \circ \psi \circ \varphi} \end{array}$$

$$\begin{array}{ccc} \coprod_{\varphi} \circ \text{Id}_{\mathbf{Set}^S} & \xrightarrow{\text{id}_{\coprod_{\varphi}} * \nu^S} & \coprod_{\varphi} \circ \coprod_{\text{id}_S} \\ \downarrow \text{id}_{\coprod_{\varphi}} & & \downarrow \gamma^{\text{id}_S, \varphi} \\ \coprod_{\varphi} & \xrightarrow{\text{id}_{\coprod_{\varphi}}} & \coprod_{\varphi \circ \text{id}_S} \end{array} \quad \begin{array}{ccc} \text{Id}_{\mathbf{Set}^T} \circ \coprod_{\varphi} & \xrightarrow{\nu^T * \text{id}_{\coprod_{\varphi}}} & \coprod_{\text{id}_T} \circ \coprod_{\varphi} \\ \downarrow \text{id}_{\coprod_{\varphi}} & & \downarrow \gamma^{\varphi, \text{id}_T} \\ \coprod_{\varphi} & \xrightarrow{\text{id}_{\coprod_{\varphi}}} & \coprod_{\text{id}_T \circ \varphi} \end{array}$$

□

From now on, when dealing with a pseudo-functor we will restrict ourselves to define explicitly only its object and morphism mappings, if about the remaining data and conditions involved in it, i.e., the natural isomorphisms and the coherence conditions, there is not any doubt.

Remark. The particular case of the just proved proposition when the sets of sorts are $S^* \times S$ and $S^* \times S^*$, and the mapping from $S^* \times S$ to $S^* \times S^*$ is

$$1 \times \check{\chi}_S \begin{cases} S^* \times S \longrightarrow S^* \times S^* \\ (w, s) \longmapsto (w, (s)) \end{cases}$$

will be specially useful in the fourth section (on Hall and Bénabou algebras) to prove the isomorphy between the free Bénabou algebra on the $S^* \times S^*$ -sorted set $\coprod_{1 \times \check{\chi}_S} \Sigma$ associated to a given $S^* \times S$ -sorted set (below called an S -sorted signature) Σ and the Bénabou algebra of terms for (S, Σ) , as well as in the fifth section (on the morphisms of Fujiwara) to provide an alternative, but equivalent, definition of the concept of morphism of Fujiwara between signatures.

Definition 1. The category \mathbf{MSet} of *many-sorted sets* and *many-sorted mappings*, obtained by applying the construction of Ehresmann-Grothendieck to the contravariant functor \mathbf{MSet} from \mathbf{Set} to \mathbf{Cat} , is $\mathbf{MSet} = \int^{\mathbf{Set}} \mathbf{MSet}$.

Therefore \mathbf{MSet} has as objects the pairs (S, A) , where S is a set and A an S -sorted set, and as morphisms from (S, A) to (T, B) the pairs (φ, f) , where $\varphi: S \longrightarrow T$ and $f: A \longrightarrow B_{\varphi}$. From now on, to shorten terminology, we will say *ms-set* and *ms-mapping* instead of *many-sorted set* and *many-sorted mapping*, respectively.

From the definition of the category \mathbf{MSet} it follows, immediately, that the projection functor $\pi_{\mathbf{MSet}}$ for \mathbf{MSet} is a split fibration (observe that, for every set S , the fiber of $\pi_{\mathbf{MSet}}$ in S is, essentially, the category \mathbf{Set}^S of S -sorted sets and mappings).

On the other hand, if we apply the construction of Ehresmann-Grothendieck to the pseudo-functor $\mathbf{MSet}^{\mathbb{I}}$, then we get a category with the same objects as

\mathbf{MSet} , but with morphisms from (S, A) to (T, B) the pairs (φ, f) , where $\varphi: S \longrightarrow T$ and $f: \coprod_{\varphi} A \longrightarrow B$. However, for every morphism $\varphi: S \longrightarrow T$, we have that $\coprod_{\varphi} \dashv \Delta_{\varphi}$, hence $\text{Hom}(\coprod_{\varphi} A, B)$ and $\text{Hom}(A, B_{\varphi})$ are naturally isomorphic, thus the categories $\int^{\mathbf{Set}} \mathbf{MSet}$ and $\int_{\mathbf{Set}} \mathbf{MSet}^{\text{II}}$ are isomorphic (observe the use, in the symbol of integration (also called the integral of Grothendieck), of the subscript to indicate the covariant situation, and of the superscript to indicate the contravariant one). From this it follows that the functor $\pi_{\mathbf{MSet}}$ is also a split opfibration. Therefore \mathbf{MSet} is a split bifibration on \mathbf{Set} .

Remark. The construction of Ehresmann-Grothendieck applied to the contravariant functor \mathbf{MSet} produces not only the category \mathbf{MSet} , but also, implicitly, a logic, the internal logic of \mathbf{MSet} (which, we recall, is the trivalued logic of Heyting), from the combination, by means of logical morphisms between the fibers of $\pi_{\mathbf{MSet}}$, of the Boolean internal logics of the just named fibers. Informally speaking, we can say that globally the category \mathbf{MSet} has an intermediate logic, but that locally (in its fibers) it is Boolean (in the same way as a manifold is a space which locally looks like \mathbb{R}^n (or \mathbb{C}^n) but not necessarily globally). Thus, in this case, we see that the system of laws governing the world obtained by synthesizing a family of given interwoven worlds, each of them governed by its proper system of laws, is not necessarily identical to anyone of the local systems of laws.

To show the bicompleteness of the category \mathbf{MSet} and of some other categories defined a bit further on, in this same section, the following definition and propositions, stated in [58] and (partially) in [29], are particularly useful, since they give sufficient conditions that are, mostly, easily verifiable for the cases we will be considering.

Definition 2. (Cf., [58], p. 249) We say that a functor $F: \mathbf{C}^{\text{op}} \longrightarrow \mathbf{Cat}$ is *locally reversible* if, for every morphism $h: c \longrightarrow d$ in \mathbf{C} , the functor $F(h)$ from $F(d)$ to $F(c)$ has a left adjoint.

Proposition 2. *Let $F: \mathbf{C}^{\text{op}} \longrightarrow \mathbf{Cat}$ be a functor. If \mathbf{C} is complete, for every object $c \in \mathbf{C}$, the category $F(c)$ is complete and, for every morphism $h: c \longrightarrow d$ in \mathbf{C} , the functor $F(h)$ from $F(d)$ to $F(c)$ is continuous (i.e., preserves projective limits), then $\int^{\mathbf{C}} F$ is complete.*

Proof. See [58], pp. 247–248. □

Proposition 3. *Let $F: \mathbf{C}^{\text{op}} \longrightarrow \mathbf{Cat}$ be a functor. If \mathbf{C} is cocomplete, for every object $c \in \mathbf{C}$, the category $F(c)$ is cocomplete, and F is locally reversible, then $\int^{\mathbf{C}} F$ is cocomplete.*

Proof. See [58], pp. 250–251. □

Corollary 1. *The category \mathbf{MSet} is bicomplete.*

Proof. The category \mathbf{MSet} is complete because \mathbf{Set} is complete, for every set S , $\mathbf{MSet}(S) = \mathbf{Set}^S$ is complete, and, for every mapping $\varphi: S \longrightarrow T$, the functor $\mathbf{MSet}(\varphi) = \Delta_{\varphi}$ from \mathbf{Set}^T to \mathbf{Set}^S is continuous, since it has \coprod_{φ} as a left adjoint.

The category \mathbf{MSet} is cocomplete because \mathbf{Set} is cocomplete, for every set S , $\mathbf{MSet}(S) = \mathbf{Set}^S$ is cocomplete, and the contravariant functor \mathbf{MSet} is locally reversible. □

Our next goal is to define the category \mathbf{Sig} , of standard many-sorted signatures and many-sorted signature morphisms, by applying the construction of Ehresmann-Grothendieck to a contravariant functor \mathbf{Sig} from \mathbf{Set} to \mathbf{Cat} . The category \mathbf{Sig}

will show to be fundamental to get, by means of the same construction, but applied to a contravariant functor from **Sig** to **Cat**, the category of many-sorted algebras, and also (as will be seen in the fifth section on morphisms of Fujiwara) to build on it, through the construction of Kleisli, another category with the same objects that **Sig**, but with a new type of morphisms, the polyderivors, which will show to be adequate to prove, in the last section, the category-theoretical equivalence between the specifications for Hall and Bénabou algebras.

Before we prove the existence of the contravariant functor **Sig** in the following proposition, we recall that, for a set of sorts S , the category of S -sorted signatures, denoted by $\mathbf{Sig}(S)$, is $\mathbf{Set}^{S^* \times S}$, where, we recall, S^* is the underlying set of the free monoid on S . Therefore an S -sorted signature is a function Σ from $S^* \times S$ to \mathcal{U} which sends a pair $(w, s) \in S^* \times S$ to the set $\Sigma_{w,s}$ of the *formal operations* of *arity* w , *sort* (or *coarity*) s , and *rank* (or *biarity*) (w, s) ; and an S -sorted signature *morphism* from Σ to Σ' an ordered triple (Σ, d, Σ') , abbreviated to $d: \Sigma \longrightarrow \Sigma'$, where d is an element of $\prod_{(w,s) \in S^* \times S} \text{Hom}(\Sigma_{w,s}, \Sigma'_{w,s})$. Thus, for $(w, s) \in S^* \times S$, $d_{w,s}$ is a mapping from $\Sigma_{w,s}$ to $\Sigma'_{w,s}$ which sends a formal operation σ in $\Sigma_{w,s}$ to the formal operation $d_{w,s}(\sigma)$, abbreviated to $d(\sigma)$, in $\Sigma'_{w,s}$. Sometimes we will write $\sigma: w \longrightarrow s$ to indicate that the formal operation σ belongs to $\Sigma_{w,s}$.

Proposition 4. *There exists a contravariant functor **Sig** from **Set** to **Cat** defined as follows*

- (1) **Sig** sends a set (of sorts) S to $\mathbf{Sig}(S) = \mathbf{Sig}(S)$, the category of S -sorted signatures.
- (2) **Sig** sends a mapping φ from S to T to the functor $\mathbf{Sig}(\varphi) = \Delta_{\varphi^* \times \varphi}$ from $\mathbf{Sig}(T)$ to $\mathbf{Sig}(S)$ which relabels T -sorted signatures into S -sorted signatures, i.e., **Sig**(φ) assigns to a T -sorted signature Λ the S -sorted signature $\mathbf{Sig}(\varphi)(\Lambda) = \Lambda_{\varphi^* \times \varphi}$, and assigns to a morphism of T -sorted signatures d from Λ to Λ' the morphism of S -sorted signatures $\mathbf{Sig}(\varphi)(d) = d_{\varphi^* \times \varphi}$ from $\Lambda_{\varphi^* \times \varphi}$ to $\Lambda'_{\varphi^* \times \varphi}$.

Proof. Because **Sig** is the composition of the covariant endofunctor $(\cdot)^* \times (\cdot)$ of **Set** which sends a set S to $S^* \times S$ and a mapping $\varphi: S \longrightarrow T$ to the mapping $\varphi^* \times \varphi: S^* \times S \longrightarrow T^* \times T$ (which assigns to (w, s) in $S^* \times S$, $(\varphi^*(w), \varphi(s))$ in $T^* \times T$), and the contravariant functor **MSet** from **Set** to **Cat**. \square

Definition 3. The category **Sig** of *many-sorted signatures* and *many-sorted signature morphisms*, obtained by applying the construction of Ehresmann-Grothendieck to the contravariant functor **Sig** on **Set** to **Cat**, is $\mathbf{Sig} = \int^{\mathbf{Set}} \mathbf{Sig}$.

Therefore the category **Sig** has as objects the pairs (S, Σ) , where S is a set of sorts and Σ an S -sorted signature and as many-sorted signature morphisms from (S, Σ) to (T, Λ) the pairs (φ, d) , where $\varphi: S \longrightarrow T$ is a morphism in **Set** while $d: \Sigma \longrightarrow \Lambda_{\varphi^* \times \varphi}$ is a morphism in $\mathbf{Sig}(S)$. The composition of

$$(\varphi, d): (S, \Sigma) \longrightarrow (T, \Lambda) \quad \text{and} \quad (\psi, e): (T, \Lambda) \longrightarrow (U, \Omega),$$

denoted by $(\psi, e) \circ (\varphi, d)$, is $(\psi \circ \varphi, e_{\varphi^* \times \varphi} \circ d)$, where

$$e_{\varphi^* \times \varphi}: \Lambda_{\varphi^* \times \varphi} \longrightarrow (\Omega_{\psi^* \times \psi})_{\varphi^* \times \varphi} = \Omega_{(\psi \circ \varphi)^* \times (\psi \circ \varphi)}.$$

From now on, unless otherwise stated, we will write Σ , Λ , Ω , and Ξ instead of (S, Σ) , (T, Λ) , (U, Ω) , and (X, Ξ) , respectively, and \mathbf{d} , \mathbf{e} , and \mathbf{h} , instead of (φ, d) , (ψ, e) , and (γ, h) , respectively. Furthermore, to shorten terminology, we will drop the qualifying adjective “many-sorted” and thus we will say *signature* and *signature morphism* instead of *many-sorted signature* and *many-sorted signature morphism*, respectively.

Remark. The category **Sig**, as was the case for **MSet**, is also a split bifibration on **Set** through the projection functor $\pi_{\mathbf{Sig}}$ for **Sig**.

Since the category **Sig** can be identified to a subcategory of the category **Sig_{po}**, defined in the fifth section, we refer to that section for examples of signature morphisms.

Proposition 5. *The category **Sig** is bicomplete.*

Proof. The proof of the bicompleteness of the category **Sig** is formally identical to that of the category **MSet**. \square

After having defined the categories $\mathbf{MSet} = \int^{\mathbf{Set}} \mathbf{MSet}$ and $\mathbf{Sig} = \int^{\mathbf{Set}} \mathbf{Sig}$ and examined some of its most useful properties (from the standpoint of general algebra), we proceed next to define the category **Alg** of many-sorted algebras by applying the construction of Ehresmann-Grothendieck to a suitable contravariant functor **Alg** defined on **Sig** and taking values in **Cat**. Besides, we prove that the category **Alg** is concrete and univocally transportable relative to a “forgetful” functor to an adequate category, that this forgetful functor, in addition, has a left adjoint, and that **Alg** is a bicomplete category.

Before we realize what has been announced we recall that, for a signature Σ and an S -sorted set A , the $S^* \times S$ -sorted set of the *finitary operations on A* , $\mathbf{HO}_{p_S}(A)$ (thus denoted because, as we will show in the fourth section, it is an example of a Hall algebra), is $(\text{Hom}(A_w, A_s))_{(w,s) \in S^* \times S}$, where $A_w = \prod_{i \in |w|} A_{w_i}$, with $|w|$ denoting the length of the word w ; and that a *structure of Σ -algebra on A* is a morphism $F = (F_{w,s})_{(w,s) \in S^* \times S}$ in $\mathbf{Sig}(S)$ from Σ to $\mathbf{HO}_{p_S}(A)$. For a pair $(w, s) \in S^* \times S$ and a formal operation $\sigma \in \Sigma_{w,s}$, in order to simplify the notation, the operation from A_w to A_s corresponding to σ under $F_{w,s}$ will be written as F_σ instead of $F_{w,s}(\sigma)$. Then the category of Σ -algebras, denoted by $\mathbf{Alg}(\Sigma)$, has as objects the pairs (A, F) , abbreviated to \mathbf{A} , where A is an S -sorted set and F a structure of Σ -algebra on A ; and as morphisms from \mathbf{A} to \mathbf{B} , where $\mathbf{B} = (B, G)$, the Σ -homomorphisms, i.e., the triples $(\mathbf{A}, f, \mathbf{B})$, abbreviated to $f: \mathbf{A} \rightarrow \mathbf{B}$, where f is an S -sorted mapping from A to B such that, for every $(w, s) \in S^* \times S$, $\sigma \in \Sigma_{w,s}$, and $(a_i)_{i \in |w|} \in A_w$ we have that

$$f_s(F_\sigma((a_i)_{i \in |w|})) = G_\sigma(f_w((a_i)_{i \in |w|})),$$

where f_w is the mapping $\prod_{i \in |w|} f_{w_i}$ from A_w to B_w which sends $(a_i)_{i \in |w|}$ in A_w to $(f_{w_i}(a_i))_{i \in |w|}$ in B_w , or, what is equivalent, such that, for every $(w, s) \in S^* \times S$ and $\sigma \in \Sigma_{w,s}$, the following diagram commutes

$$\begin{array}{ccc} A_w & \xrightarrow{f_w} & B_w \\ F_\sigma \downarrow & & \downarrow G_\sigma \\ A_s & \xrightarrow{f_s} & B_s \end{array}$$

Sometimes, to avoid any confusion, we will denote the structures of Σ -algebra of the Σ -algebras $\mathbf{A}, \mathbf{B}, \dots$, by $F^{\mathbf{A}}, F^{\mathbf{B}}, \dots$, respectively, and the components of $F^{\mathbf{A}}, F^{\mathbf{B}}, \dots$, as $F_\sigma^{\mathbf{A}}, F_\sigma^{\mathbf{B}}, \dots$, respectively.

Proposition 6. *There exists a contravariant functor **Alg** from **Sig** to **Cat** which sends a signature Σ to $\mathbf{Alg}(\Sigma) = \mathbf{Alg}(\Sigma)$, the category of Σ -algebras, and a signature morphism $\mathbf{d}: \Sigma \rightarrow \Lambda$ to the functor $\mathbf{Alg}(\mathbf{d}) = \mathbf{d}^*: \mathbf{Alg}(\Lambda) \rightarrow \mathbf{Alg}(\Sigma)$ defined as follows*

- (1) \mathbf{d}^* assigns to a $\mathbf{\Lambda}$ -algebra $\mathbf{B} = (B, G)$ the $\mathbf{\Sigma}$ -algebra $\mathbf{d}^*(\mathbf{B}) = (B_\varphi, G^{\mathbf{d}})$, where $G^{\mathbf{d}}$ is the composition of the $S^* \times S$ -sorted mappings

$$d: \Sigma \longrightarrow \Lambda_{\varphi^* \times \varphi} \quad \text{and} \quad G_{\varphi^* \times \varphi}: \Lambda_{\varphi^* \times \varphi} \longrightarrow \text{HOp}_T(B)_{\varphi^* \times \varphi}.$$

We agree that, for a formal operation $\sigma \in \Sigma_{w,s}$, $G_{d(\sigma)}: B_{\varphi^*(w)} \longrightarrow B_{\varphi(s)}$ denotes the value of $G^{\mathbf{d}}$ at σ .

- (2) \mathbf{d}^* assigns to a $\mathbf{\Lambda}$ -homomorphism f from \mathbf{B} to \mathbf{B}' the $\mathbf{\Sigma}$ -homomorphism $\mathbf{d}^*(f) = f_\varphi$ from $\mathbf{d}^*(\mathbf{B})$ to $\mathbf{d}^*(\mathbf{B}')$.

Proof. For every $\mathbf{\Lambda}$ -algebra (B, G) , we have that $G: \Lambda \longrightarrow \text{HOp}_T(B)$. Then, by composing $d: \Sigma \longrightarrow \Lambda_{\varphi^* \times \varphi}$ and $G_{\varphi^* \times \varphi}: \Lambda_{\varphi^* \times \varphi} \longrightarrow \text{HOp}_T(B)_{\varphi^* \times \varphi}$, and taking into account that $\text{HOp}_T(B)_{\varphi^* \times \varphi} = \text{HOp}_S(B_\varphi)$, we have that $G^{\mathbf{d}} = G_{\varphi^* \times \varphi} \circ d$ is a structure of $\mathbf{\Sigma}$ -algebra on B_φ .

On the other hand, given $(w, s) \in S^* \times S$ and $\sigma \in \Sigma_{w,s}$, $d(\sigma) \in \Lambda_{\varphi^*(w), \varphi(s)}$, thus, being f a $\mathbf{\Lambda}$ -homomorphism from (B, G) to (B', G') , we have that $f_{\varphi(s)} \circ G_{d(\sigma)} = G'_{d(\sigma)} \circ f_{\varphi^*(w)}$, therefore, because $G^{\mathbf{d}} = G_{d(\sigma)}$ and $G'^{\mathbf{d}} = G'_{d(\sigma)}$, $(f_\varphi)_s \circ G^{\mathbf{d}} = G'^{\mathbf{d}} \circ (f_\varphi)_w$, hence f_φ is a $\mathbf{\Sigma}$ -homomorphism from $(B_\varphi, G^{\mathbf{d}})$ to $(B'_\varphi, G'^{\mathbf{d}})$.

Since the identities and the composites are, obviously, preserved by \mathbf{d}^* , it follows that \mathbf{d}^* is a functor from $\mathbf{Alg}(\mathbf{\Lambda})$ to $\mathbf{Alg}(\mathbf{\Sigma})$. \square

Definition 4. The category \mathbf{Alg} of *many-sorted algebras* and *many-sorted algebra homomorphisms*, obtained by applying the construction of Ehresmann-Grothendieck to the contravariant functor \mathbf{Alg} from \mathbf{Sig} to \mathbf{Cat} , is $\mathbf{Alg} = \int^{\mathbf{Sig}} \mathbf{Alg}$.

Therefore the category \mathbf{Alg} has as objects the pairs $(\mathbf{\Sigma}, \mathbf{A})$, where $\mathbf{\Sigma}$ is a signature and \mathbf{A} a $\mathbf{\Sigma}$ -algebra, and as morphisms from $(\mathbf{\Sigma}, \mathbf{A})$ to $(\mathbf{\Lambda}, \mathbf{B})$, the pairs (\mathbf{d}, f) , with \mathbf{d} a signature morphism from $\mathbf{\Sigma}$ to $\mathbf{\Lambda}$ and f a $\mathbf{\Sigma}$ -homomorphism from \mathbf{A} to $\mathbf{d}^*(\mathbf{B})$. Hence, for every $(w, s) \in S^* \times S$ and $\sigma \in \Sigma_{w,s}$ the following diagram commutes

$$\begin{array}{ccc} A_w & \xrightarrow{f_w} & B_{\varphi^*(w)} \\ F_\sigma \downarrow & & \downarrow G_{d(\sigma)} \\ A_s & \xrightarrow{f_s} & B_{\varphi(s)} \end{array}$$

From now on, to shorten terminology, we will say *algebra* and *algebra homomorphism*, or, simply, *homomorphism*, instead of *many-sorted algebra* and *many-sorted algebra homomorphism*, respectively. Sometimes, to avoid any confusion, we denote an algebra $(\mathbf{\Sigma}, \mathbf{A})$ and an homomorphism (\mathbf{d}, f) also by (S, Σ, A, F) and (φ, d, f) , respectively.

Since the category \mathbf{Alg} can be identified to a subcategory of the category \mathbf{Alg}_{pd} , defined in the fifth section, we refer to that section for examples of homomorphisms between algebras.

Proposition 7. *The category \mathbf{Alg} is a concrete and univocally transportable category.*

Proof. It is enough to specify a functor from \mathbf{Alg} to a convenient category of sorted sets labelled by signatures.

Let $G_{\mathbf{MSet}}$ be the forgetful functor from \mathbf{Alg} to \mathbf{MSet} (that is not a fibration), $\pi_{\mathbf{Alg}}$ the projection functor for \mathbf{Alg} , and $(\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}, (P_0, P_1))$ the pullback of the projection functors $\pi_{\mathbf{MSet}}$ and $\pi_{\mathbf{Sig}}$, for \mathbf{MSet} and \mathbf{Sig} , respectively, where

- (1) The category $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ has as objects, essentially, triples (S, Σ, A) , with (S, Σ) a signature and A an S -sorted set, and as morphisms from (S, Σ, A) to (T, Λ, B) triples (φ, d, f) , such that (φ, d) is a signature morphism from (S, Σ) to (T, Λ) and (φ, f) a mapping from (S, A) to (T, B) , while
- (2) P_0 is the functor from $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ to \mathbf{MSet} which sends a morphism (φ, d, f) from (S, Σ, A) to (T, Λ, B) to the ms-mapping (φ, f) from (S, A) to (T, B) , and P_1 is the functor from $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ to \mathbf{Sig} which sends a morphism (φ, d, f) from (S, Σ, A) to (T, Λ, B) to the signature morphism (φ, d) from (S, Σ) to (T, Λ) .

Then we have that the structural functors P_0 and P_1 are fibrations, and that the unique functor G from \mathbf{Alg} to $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ such that $P_0 \circ G = G_{\mathbf{MSet}}$ and $P_1 \circ G = \pi_{\mathbf{Alg}}$, as in the following diagram

$$\begin{array}{ccccc}
 \mathbf{Alg} & & & & \\
 \downarrow G & \searrow \pi_{\mathbf{Alg}} & & & \\
 \mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig} & \xrightarrow{P_1} & \mathbf{Sig} & & \\
 \downarrow P_0 & & \downarrow \pi_{\mathbf{Sig}} & & \\
 \mathbf{MSet} & \xrightarrow{\pi_{\mathbf{MSet}}} & \mathbf{Set} & & \\
 \uparrow G_{\mathbf{MSet}} & & & &
 \end{array}$$

makes the category \mathbf{Alg} a concrete and univocally transportable category on the category $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$. \square

The functor G from \mathbf{Alg} to $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ not only has the above mentioned properties, but it also has a left adjoint, obtained from the family $(\mathbf{T}_{\Sigma})_{\Sigma \in \mathbf{Sig}}$ where, for a signature Σ in \mathbf{Sig} , the functor \mathbf{T}_{Σ} from \mathbf{Set}^S to $\mathbf{Alg}(\Sigma)$ is the left adjoint to the forgetful functor G_{Σ} from $\mathbf{Alg}(\Sigma)$ to \mathbf{Set}^S . And this left adjoint to G , that, as we will see below, allows us to get translations between free algebras, will be used, once defined the many-sorted equations in the third section, to translate, for a signature morphism, many-sorted equations for the source signature to many-sorted equations for the target signature. This translation of equations, together with the invariant character of the relation of satisfaction under change of notation, will allow us to define, also in the third section, the many-sorted equational institution (more general than that defined by Goguen and Burstall in [25]) that embodies the essentials of semantical many-sorted equational deduction.

Before we prove the existence of a left adjoint to G , we recall that

- (1) For a signature Σ and an S -sorted set of variables X , $\mathbf{T}_{\Sigma}(X)$, the value of the functor \mathbf{T}_{Σ} in X , is the *free* (also called the *term* or *word*) Σ -algebra on X , and η_X is the *insertion (of the generators) X into $\mathbf{T}_{\Sigma}(X)$* , the underlying S -sorted set of $\mathbf{T}_{\Sigma}(X)$;
- (2) For a Σ -algebra \mathbf{A} and a *valuation f of the S -sorted set of variables X in \mathbf{A}* , i.e., an S -sorted mapping f from X to \mathbf{A} , f^{\sharp} denotes the *canonical extension of f up to $\mathbf{T}_{\Sigma}(X)$* , i.e., the unique Σ -homomorphism from $\mathbf{T}_{\Sigma}(X)$ to \mathbf{A} such that $f^{\sharp} \circ \eta_X = f$; and
- (3) For an S -sorted mapping f from X to Y , f^{\circledast} denotes the unique Σ -homomorphism from $\mathbf{T}_{\Sigma}(X)$ to $\mathbf{T}_{\Sigma}(Y)$ such that $f^{\circledast} \circ \eta_X = \eta_Y \circ f$, i.e., the value of the functor \mathbf{T}_{Σ} in f . Therefore f^{\circledast} is also $(\eta_Y \circ f)^{\sharp}$.

Moreover, transposing to the many-sorted case the terminology coined for the single-sorted case, we call, for $s \in S$, the elements of $\mathbf{T}_\Sigma(X)_s$, *many-sorted terms for Σ of type (X, s)* , from now on abbreviated to *terms for Σ of type (X, s)* , or, simply, to *terms of type (X, s)* . We point out that what we have called terms for Σ of type (X, s) are also known, for those following the terminology in Grätzer [28], p. 39, as *polynomial symbols (for Σ) of type (X, s)* .

Proposition 8. *There exists a functor $\mathbf{T}: \mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig} \longrightarrow \mathbf{Alg}$ left adjoint to the functor $\mathbf{G}: \mathbf{Alg} \longrightarrow \mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$.*

Proof. The functor \mathbf{T} from $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$ to \mathbf{Alg} given on objects (S, Σ, X) by $\mathbf{T}(S, \Sigma, X) = (\Sigma, \mathbf{T}_\Sigma(X))$ and on arrows $(\varphi, d, f): (S, \Sigma, X) \longrightarrow (T, \Lambda, Y)$ as

$$(\mathbf{d}, f^{\mathbf{d}}): (\Sigma, \mathbf{T}_\Sigma(X)) \longrightarrow (\Lambda, \mathbf{T}_\Lambda(Y)),$$

where $f^{\mathbf{d}} = ((\eta_Y)_\varphi \circ f)^\sharp$ is the canonical extension of the S -sorted mapping $(\eta_Y)_\varphi \circ f$ from X to $\mathbf{T}_\Lambda(Y)_\varphi$ up to the free Σ -algebra on X , is left adjoint to the functor \mathbf{G} . \square

For a morphism $(\varphi, d, f): (S, \Sigma, X) \longrightarrow (T, \Lambda, Y)$ in $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, the functor $\mathbf{T}: \mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig} \longrightarrow \mathbf{Alg}$ acting on (φ, d, f) allows us to get the Σ -homomorphism $f^{\mathbf{d}}$ from $\mathbf{T}_\Sigma(X)$ to $\mathbf{T}_\Lambda(Y)_\varphi$, hence, for $s \in S$, it translates terms for Σ of type (X, s) , i.e., elements P of $\mathbf{T}_\Sigma(X)_s$, into terms for Λ of type $(Y, \varphi(s))$, i.e., elements $f_s^{\mathbf{d}}(P)$ of $\mathbf{T}_\Lambda(Y)_{\varphi(s)}$.

In particular, the unit η^φ of the adjunction $\coprod_\varphi \dashv \Delta_\varphi$ provides, for every S -sorted set X , the S -sorted mapping $\eta_X^\varphi: X \longrightarrow (\coprod_\varphi X)_\varphi$ and if $\mathbf{d}: \Sigma \longrightarrow \Lambda$ is a morphism of signatures, then $(\varphi, d, \eta_X^\varphi): (S, \Sigma, X) \longrightarrow (T, \Lambda, \coprod_\varphi X)$ is a morphism in $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, hence the functor \mathbf{T} acting on $(\varphi, d, \eta_X^\varphi)$ gives rise to the morphism

$$(\mathbf{d}, \eta_X^{\mathbf{d}}): (\Sigma, \mathbf{T}_\Sigma(X)) \longrightarrow (\Lambda, \mathbf{T}_\Lambda(\coprod_\varphi X)),$$

where $\eta_X^{\mathbf{d}} = ((\eta_{\coprod_\varphi X})_\varphi \circ \eta_X^\varphi)^\sharp$ is the Σ -homomorphism from $\mathbf{T}_\Sigma(X)$ to $\mathbf{T}_\Lambda(\coprod_\varphi X)_\varphi$ that extends the S -sorted mapping $(\eta_{\coprod_\varphi X})_\varphi \circ \eta_X^\varphi$ from X to $\mathbf{T}_\Lambda(\coprod_\varphi X)_\varphi$. Therefore, for $s \in S$, $\eta_{X,s}^{\mathbf{d}}$, the s -th component of $\eta_X^{\mathbf{d}}$, translates terms for Σ of type (X, s) into terms for Λ of type $(\coprod_\varphi X, \varphi(s))$. The Σ -homomorphisms $\eta_X^{\mathbf{d}}$, as stated in the following proposition, are in fact the components of a natural transformation, and this contributes to explain their relevance as translators.

Proposition 9. *Let \mathbf{d} be a morphism of signatures from Σ to Λ . Then the family $\eta^{\mathbf{d}} = (\eta_X^{\mathbf{d}})_{X \in \mathcal{U}}$, which to an S -sorted set X assigns the Σ -homomorphism $\eta_X^{\mathbf{d}}$ from $\mathbf{T}_\Sigma(X)$ to $\mathbf{T}_\Lambda(\coprod_\varphi X)_\varphi$, is a natural transformation from \mathbf{T}_Σ to $\mathbf{d}^* \circ \mathbf{T}_\Lambda \circ \coprod_\varphi$, and so, for the forgetful functor \mathbf{G}_Σ from $\mathbf{Alg}(\Sigma)$ to \mathbf{Set}^S , the family $\mathbf{G}_\Sigma * \eta^{\mathbf{d}}$, i.e., the horizontal composition of the natural transformation $\eta^{\mathbf{d}}$ and \mathbf{G}_Σ , also denoted by $\eta^{\mathbf{d}}$, is a natural transformation from $\mathbf{T}_\Sigma = \mathbf{G}_\Sigma \circ \mathbf{T}_\Sigma$ to $\Delta_\varphi \circ \mathbf{T}_\Lambda \circ \coprod_\varphi$, since $\mathbf{G}_\Sigma \circ \mathbf{d}^* = \Delta_\varphi \circ \mathbf{G}_\Lambda$ and $\mathbf{T}_\Lambda = \mathbf{G}_\Lambda \circ \mathbf{T}_\Lambda$.*

Proof. It follows after the commutativity of the following diagram

$$\begin{array}{ccccc}
 & & T_{\Sigma}(X) & \xrightarrow{f^{\circledast}} & T_{\Sigma}(Y) \\
 & \nearrow \eta_X & \downarrow \eta_X^{\mathbf{d}} & & \downarrow \eta_Y^{\mathbf{d}} \\
 X & \xrightarrow{f} & Y & & \\
 \downarrow \eta_X^{\circ} & & \downarrow \eta_Y^{\circ} & & \\
 T_{\Lambda}(\coprod_{\varphi} X)_{\varphi} & \xrightarrow{(\coprod_{\varphi} f)^{\circledast}} & T_{\Lambda}(\coprod_{\varphi} Y)_{\varphi} & & \\
 \downarrow (\eta_{\coprod_{\varphi} X})_{\varphi} & & \downarrow (\eta_{\coprod_{\varphi} Y})_{\varphi} & & \\
 (\coprod_{\varphi} X)_{\varphi} & \xrightarrow{(\coprod_{\varphi} f)_{\varphi}} & (\coprod_{\varphi} Y)_{\varphi} & &
 \end{array}$$

□

Remark. The natural transformation $\eta^{\mathbf{d}} = G_{\Sigma} * \eta^{\mathbf{d}}$ from the functor T_{Σ} to the functor $\Delta_{\varphi} \circ T_{\Lambda} \circ \coprod_{\varphi}$, will be used later on, in this same section, to provide, for a signature morphism $\mathbf{d}: \Sigma \rightarrow \Lambda$, an alternative, but equivalent, definition of (the morphism mapping of) a translation functor \mathbf{d}_{\circ} from $\mathbf{Ter}(\Sigma)$ to $\mathbf{Ter}(\Lambda)$, where, for a signature Σ , we anticipate, $\mathbf{Ter}(\Sigma)$ is the category with objects the S -sorted sets and morphisms from X to Y the S -sorted mappings from Y to $T_{\Sigma}(X)$, that we will call terms for Σ of type (X, Y) .

Before proceeding to prove the bicompleteness of the category \mathbf{Alg} , we make next a brief excursus about the utility of some big subcategories of \mathbf{Alg} . As we know, from \mathbf{Alg} to \mathbf{Sig} we have the fibration $\pi_{\mathbf{Alg}}$ and from \mathbf{Sig} to \mathbf{Set} the fibration $\pi_{\mathbf{Sig}}$, hence from \mathbf{Alg} to \mathbf{Set} , by composing both fibrations, we get the fibration $\pi_{\mathbf{Sig}, \mathbf{Alg}}$. And this composed fibration allows us to get, for every set of sorts S , the corresponding fiber $\mathbf{Alg}(S)$, that we call the category of S -algebras. We recall that $\mathbf{Alg}(S)$ has, essentially, as objects the pairs (Σ, \mathbf{A}) , where Σ is an S -sorted signature and \mathbf{A} a Σ -algebra, and as morphisms from (Σ, \mathbf{A}) to (Λ, \mathbf{B}) , the pairs (d, f) , with d an S -sorted signature morphism from Σ to Λ and f a Σ -homomorphism from \mathbf{A} to $d^*(\mathbf{B})$.

At first glance, the categories of the type $\mathbf{Alg}(S)$ can appear to be unnecessarily or excessively general, hence almost useless. However, they show themselves to be useful, for example, to get a full understanding, i.e., to complete category-theoretically the explanation, of some classical constructions in universal algebra as, e.g., those due to Birkhoff-Frink in [3], where they state, among others interesting results, the following representation theorems:

- (1) Let (A, J) be an algebraic closure space. Then there exists a single-sorted signature $\Sigma^{(A, J)}$ and a structure of $\Sigma^{(A, J)}$ -algebra $F^{(A, J)}$ on A such that

$$(A, \text{Sg}_{(A, F^{(A, J)})}) = (A, J),$$

where $\text{Sg}_{(A, F^{(A, J)})}$ is the generated subalgebra operator on A induced by the $\Sigma^{(A, J)}$ -algebra $(A, F^{(A, J)})$ (see [3], p. 300).

- (2) Let \mathbf{L} be a lattice. Then \mathbf{L} is an algebraic lattice, i.e., it is, besides, complete and compactly generated (see for the meaning of these terms, e.g., [11], p. 17), iff there exists a single-sorted signature Σ and a Σ -algebra \mathbf{A} such that \mathbf{L} is isomorphic to the algebraic lattice determined by the fixed points of the operator $\text{Sg}_{\mathbf{A}}$ (see [3], p. 302).

To show the role that the fibers of the functor $\pi_{\mathbf{Sig}, \mathbf{Alg}}$ play in the categorization of the first theorem of Birkhoff-Frink, and because it is stated in terms of ordinary

sets, we should consider the category $\mathbf{Alg}(1)$, i.e., the fiber of $\pi_{\text{Sig,Alg}}$ in 1 , the standard final set. We recall that $\mathbf{Alg}(1)$ has as objects, essentially, the pairs (Σ, \mathbf{A}) , where $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ is a single-sorted signature, i.e., an object of $\mathbf{Set}^{\mathbb{N}}$, and $\mathbf{A} = (A, F)$ a Σ -algebra, i.e., an ordinary set A together with an \mathbb{N} -sorted mapping F from Σ to $(\text{Hom}(A^n, A))_{n \in \mathbb{N}}$, and as morphisms from (Σ, \mathbf{A}) to (Λ, \mathbf{B}) , where $\mathbf{B} = (B, G)$, the pairs (d, f) with $d = (d_n)_{n \in \mathbb{N}}$ an \mathbb{N} -sorted mapping from Σ to Λ in $\mathbf{Set}^{\mathbb{N}}$, and f a Σ -homomorphism from $\mathbf{A} = (A, F)$ to $\mathbf{B}^d = (B, G \circ d)$.

To attain the aim just stated, we should also consider the category \mathbf{ACISp} with objects the algebraic closure spaces, i.e., the pairs (A, J) , where A is an ordinary set and J an algebraic closure operator on A , and morphisms from (A, J) to (B, K) the ordered triples $((A, J), f, (B, K))$, abbreviated to $f: (A, J) \twoheadrightarrow (B, K)$, where f is a mapping from A to B such that, for every $X \subseteq A$, $f[J(X)] \subseteq K[f(X)]$.

Next, let Sg be the functor from $\mathbf{Alg}(1)$ to \mathbf{ACISp} which sends an algebra (Σ, \mathbf{A}) to the algebraic closure space $(A, \text{Sg}_{\mathbf{A}})$ where $\text{Sg}_{\mathbf{A}}$ is the generated subalgebra operator on A induced by \mathbf{A} ; and a morphism (d, f) from (Σ, \mathbf{A}) to (Λ, \mathbf{B}) to the morphism f from $(A, \text{Sg}_{\mathbf{A}})$ to $(B, \text{Sg}_{\mathbf{B}})$.

The action of Sg on the morphisms is well defined because, for every $X \subseteq A$, $f[\text{Sg}_{\mathbf{A}}(X)] \subseteq \text{Sg}_{\mathbf{B}}(f[X])$, taking into account that f is a Σ -homomorphism from $\mathbf{A} = (A, F)$ to $\mathbf{B}^d = (B, G \circ d)$, and recalling that, for an arbitrary Σ -algebra \mathbf{C} and subset Z of C , $\text{Sg}_{\mathbf{C}}(Z) = \bigcup_{n \in \mathbb{N}} \text{E}_{\mathbf{C}}^n(Z)$, where $(\text{E}_{\mathbf{C}}^n(Z))_{n \in \mathbb{N}}$ is the family in $\text{Sub}(C)$ defined recursively as: $\text{E}_{\mathbf{C}}^0(Z) = Z$, and, for $n \geq 0$, $\text{E}_{\mathbf{C}}^{n+1}(Z) = \text{E}_{\mathbf{C}}(\text{E}_{\mathbf{C}}^n(Z))$, where $\text{E}_{\mathbf{C}}$ is the operator on $\text{Sub}(C)$ which sends a subset Z of C to

$$Z \cup \left(\bigcup_{n \in \mathbb{N}} \bigcup_{\sigma \in \Sigma_n} F_{\sigma}[Z^n] \right).$$

Then the functor Sg from $\mathbf{Alg}(1)$ to \mathbf{ACISp} is surjective on the objects (and this is the first theorem of Birkhoff-Frink) and on the morphisms. In fact, given an algebraic closure space (A, J) , for the ordinary signature $\Sigma^{(A, J)} = (\Sigma_n^{(A, J)})_{n \in \mathbb{N}}$ defined, for $n \in \mathbb{N}$, as follows

$$\Sigma_n^{(A, J)} = \bigcup_{x \in A^n} (\{x\} \times J(\text{Im}(x))),$$

where $\text{Im}(x)$ is the image of the mapping $x: n \longrightarrow A$, and the Σ -algebra $(A, F^{(A, J)})$, where $F^{(A, J)}$, the structure of $\Sigma^{(A, J)}$ -algebra on A , is defined, for $n \in \mathbb{N}$, $x \in A^n$, and $a \in J(\text{Im}(x))$, as follows

$$F_{x, a}^{(A, J)} \left\{ \begin{array}{l} A^n \longrightarrow A \\ (y_0, \dots, y_{n-1}) \longmapsto \begin{cases} a, & \text{if } (y_0, \dots, y_{n-1}) = x; \\ y_0, & \text{if } (y_0, \dots, y_{n-1}) \neq x, \end{cases} \end{array} \right.$$

we have that $J = \text{Sg}_{(A, F^{(A, J)})}$, i.e., the object mapping of the functor Sg is surjective.

On the other hand, given a morphism f from an algebraic closure space (A, J) to another (B, K) , we have that the pair (d_f, f) , where d_f is the morphism from $\Sigma^{(A, J)}$ to $\Sigma^{(B, K)}$ which to a pair (x, a) , with $x \in A^n$, for some $n \in \mathbb{N}$, and $a \in J(\text{Im}(x))$, assigns the pair $(f^n(x), f(a))$, is a homomorphism from the algebra $(\Sigma^{(A, J)}, (A, F^{(A, J)}))$ to the algebra $(\Sigma^{(B, K)}, (B, F^{(B, K)}))$, and, obviously, it is sent to the morphism f by the functor Sg . Therefore the morphism mapping of Sg is surjective.

To succeed in the categorization of the second theorem of Birkhoff-Frink, we should take into consideration the category \mathbf{ALat}_{\wedge} with objects the algebraic lattices, and morphisms the lattice morphisms which preserve arbitrary infima.

Next, let Fix be the contravariant functor from \mathbf{ACISp} to \mathbf{ALat}_{\wedge} which assigns to an algebraic closure space (A, J) the algebraic lattice $\mathbf{Fix}(J)$ of the fixed points of J , and to a morphism f from (A, J) to (B, K) associates the morphism $f^{-1}[\cdot]$

(which sends a fixed point $Y = K(Y)$ of K to $f^{-1}[Y]$, its inverse image under f) from $\mathbf{Fix}(K)$ to $\mathbf{Fix}(J)$.

Then we have that \mathbf{Fix} is essentially surjective. Therefore the contravariant functor $\mathbf{Fix} \circ \mathbf{Sg}$ from $\mathbf{Alg}(1)$ to \mathbf{ALat}_\wedge is essentially surjective.

Remark. Results similar to the just stated about the theorems of Birkhoff-Frink, are also valid for a set of sorts S with two or more sorts, but replacing the category \mathbf{ACISp} by the category $\mathbf{UACISp}(S)$, of uniform algebraic S -closure spaces, as in [12].

Additional examples of the utility of the categories of the type $\mathbf{Alg}(S)$ are the category \mathbf{Mod}_r of all (right) modules over all rings (\mathbf{Mod} is fibered over \mathbf{Rng} , the fiber over each ring \mathbf{R} being the category \mathbf{Mod}_R of right \mathbf{R} -modules), and, generally speaking, all those categories that can be obtained, essentially, in the same way as was obtained \mathbf{Mod}_r , i.e., starting from some action of a mathematical construct on another one.

The category \mathbf{Alg} of algebras, as was the case for the categories \mathbf{MSet} and \mathbf{Sig} , is also complete and cocomplete. These results are already known, although we are not aware of a suitably explicit and direct proof, as that provided by us below, of the cocompleteness of \mathbf{Alg} , in what has to do, particularly, with the existence of a certain left adjoint.

Proposition 10. *The category \mathbf{Alg} is complete.*

Proof. Let $\mathbf{d}: \Sigma \longrightarrow \Lambda$ be a signature morphism. Since the forgetful functors G_Σ and G_Λ create projective limits and the following diagram commutes

$$\begin{array}{ccc} \mathbf{Alg}(\Sigma) & \xrightarrow{G_\Sigma} & \mathbf{Set}^S \\ \mathbf{d}^* \uparrow & & \uparrow \Delta_\varphi \\ \mathbf{Alg}(\Lambda) & \xrightarrow{G_\Lambda} & \mathbf{Set}^T \end{array}$$

the functor \mathbf{d}^* preserves projective limits, i.e., is continuous. But the category \mathbf{Sig} is complete, and, for every signature Σ , $\mathbf{Alg}(\Sigma)$ is complete. Therefore, by Proposition 2, the category \mathbf{Alg} is complete. \square

To prove that the category \mathbf{Alg} is cocomplete we begin by proving that, for every signature morphism $\mathbf{d}: \Sigma \longrightarrow \Lambda$, the functor \mathbf{d}^* has a left adjoint \mathbf{d}_* .

Proposition 11. *Let $\mathbf{d}: \Sigma \longrightarrow \Lambda$ be a signature morphism. Then there exists a functor $\mathbf{d}_*: \mathbf{Alg}(\Sigma) \longrightarrow \mathbf{Alg}(\Lambda)$ that is left adjoint to the functor \mathbf{d}^* .*

Proof. We begin by defining the action \mathbf{d}_* on the objects. Let \mathbf{A} be a Σ -algebra. Then $\mathbf{d}_*(\mathbf{A})$ is the Λ -algebra defined as $\mathbf{T}_\Lambda(\coprod_\varphi A)/\bar{R}^\mathbf{A}$, where $\bar{R}^\mathbf{A}$ is the congruence on $\mathbf{T}_\Lambda(\coprod_\varphi A)$ generated by the T -sorted relation $R^\mathbf{A}$, defined, for every $t \in T$, as

$$R_t^\mathbf{A} = \left\{ \left((F_\sigma^\mathbf{A}(a_i \mid i \in |w|), s), d(\sigma)((a_i, w_i) \mid i \in |w|) \right) \mid \begin{array}{l} s \in \varphi^{-1}[t], w \in S^*, \\ \sigma \in \Sigma_{w,s}, a \in A_w \end{array} \right\}.$$

Following this we define the action of \mathbf{d}_* on the morphisms. Let f be a Σ -homomorphism from \mathbf{A} to \mathbf{A}' . Then $R^\mathbf{A} \subseteq \text{Ker}(\text{pr}_{\bar{R}^{\mathbf{A}'}} \circ (\coprod_\varphi f)^\circledast)$ because, for $t \in T$ and $((F_\sigma^\mathbf{A}(a_i \mid i \in |w|), s), d(\sigma)((a_i, w_i) \mid i \in |w|)) \in R_t^\mathbf{A}$, we have that

$$\begin{aligned} [(\coprod_\varphi f)^\circledast(F_\sigma^\mathbf{A}(a_i \mid i \in |w|), s)] &= [(f_s(F_\sigma^\mathbf{A}(a_i \mid i \in |w|)), s)] \\ &= [d(\sigma)(f_{w_i}(a_i, w_i) \mid i \in |w|)] \\ &= [(\coprod_\varphi f)^\circledast(d(\sigma)((a_i, w_i) \mid i \in |w|))]. \end{aligned}$$

From this it follows that there exists a unique $\mathbf{\Lambda}$ -homomorphism $\mathbf{d}_*(f)$ from $\mathbf{d}_*(\mathbf{A})$ to $\mathbf{d}_*(\mathbf{A}')$ such that $\mathbf{d}_*(f) \circ \text{pr}_{\overline{R^{\mathbf{A}}}} = \text{pr}_{\overline{R^{\mathbf{A}'}}} \circ (\coprod_{\varphi} f)^{\circledast}$.

After this we prove that the functor \mathbf{d}_* , which sends a Σ -algebra \mathbf{A} to the $\mathbf{\Lambda}$ -algebra $\mathbf{d}_*(\mathbf{A}) = \mathbf{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} A)/\overline{R^{\mathbf{A}}}$ and a Σ -homomorphism f from \mathbf{A} to \mathbf{A}' to the $\mathbf{\Lambda}$ -homomorphism $\mathbf{d}_*(f)$ from $\mathbf{d}_*(\mathbf{A})$ to $\mathbf{d}_*(\mathbf{A}')$, is left adjoint to \mathbf{d}^* .

Let \mathbf{A} be a Σ -algebra. Then we denote by $\eta^{\mathbf{A}}$ the S -sorted mapping from A to the underlying S -sorted set of the Σ -algebra $\mathbf{d}^*(\mathbf{d}_*(\mathbf{A}))$, obtained as the inverse image under θ^{φ} (the natural isomorphism of the adjunction $\coprod_{\varphi} \dashv \Delta_{\varphi}$) of the T -sorted mapping

$$\coprod_{\varphi} A \xrightarrow{\eta_{\coprod_{\varphi} A}} \mathbf{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} A) \xrightarrow{\text{pr}_{\overline{R^{\mathbf{A}}}}} \mathbf{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} A)/\overline{R^{\mathbf{A}}}.$$

The S -sorted mapping $\eta^{\mathbf{A}}$ is a Σ -homomorphism from \mathbf{A} to $\mathbf{d}^*(\mathbf{d}_*(\mathbf{A}))$. Let $\sigma: w \longrightarrow s$ be a formal operation and $a \in A_w$, then we have that

$$\eta_s^{\mathbf{A}}(F_{\sigma}^{\mathbf{A}}(a_i \mid i \in |w|)) = [(F_{\sigma}^{\mathbf{A}}(a_i \mid i \in |w|), s)],$$

and, on the other hand, we also have that

$$\begin{aligned} F_{\sigma}^{\mathbf{d}^*(\mathbf{d}_*(\mathbf{A}))}(\eta_{w_i}^{\mathbf{A}}(a_i) \mid i \in |w|) &= F_{d(\sigma)}^{\mathbf{d}_*(\mathbf{A})}([(a_i, w_i) \mid i \in |w|]) \\ &= [F_{d(\sigma)}^{\mathbf{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} A)}((a_i, w_i) \mid i \in |w|)] \\ &= [d(\sigma)((a_i, w_i) \mid i \in |w|)], \end{aligned}$$

but, by the definition of $R_{\varphi(s)}^{\mathbf{A}}$, $[(F_{\sigma}^{\mathbf{A}}(a_i \mid i \in |w|), s)] = [d(\sigma)((a_i, w_i) \mid i \in |w|)]$. Therefore we can assert that

$$\eta_s^{\mathbf{A}}(F_{\sigma}^{\mathbf{A}}(a_i \mid i \in |w|)) = F_{\sigma}^{\mathbf{d}^*(\mathbf{d}_*(\mathbf{A}))}(\eta_{w_i}^{\mathbf{A}}(a_i) \mid i \in |w|).$$

Finally, we prove the universal property. For this, let \mathbf{B} be a $\mathbf{\Lambda}$ -algebra, f a Σ -homomorphism from \mathbf{A} to $\mathbf{d}^*(\mathbf{B})$, and \widehat{f} the T -sorted mapping associated to the S -sorted mapping $f: A \longrightarrow B_{\varphi}$. Then there exists a unique $\mathbf{\Lambda}$ -homomorphism f^{\sharp} such that the right triangle in the following diagram commutes

$$\begin{array}{ccccc} \coprod_{\varphi} A & \xrightarrow{\eta_{\coprod_{\varphi} A}} & \mathbf{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} A) & \xrightarrow{\text{pr}_{\overline{R^{\mathbf{A}}}}} & \mathbf{T}_{\mathbf{\Lambda}}(\coprod_{\varphi} A)/\overline{R^{\mathbf{A}}} \\ & \searrow \widehat{f} & \downarrow \widehat{f}^{\sharp} & \swarrow f^{\sharp} & \\ & & B & & \end{array}$$

since, for every $t \in T$, from $((F_{\sigma}^{\mathbf{A}}(a_i \mid i \in |w|), s), d(\sigma)((a_i, w_i) \mid i \in |w|)) \in R_t^{\mathbf{A}}$ it follows that

$$\begin{aligned} \widehat{f}_t^{\sharp}(F_{\sigma}^{\mathbf{A}}(a_i \mid i \in |w|), s) &= f_s^{\mathbf{A}}(F_{\sigma}^{\mathbf{A}}(a_i \mid i \in |w|)) \\ &= F_{d(\sigma)}^{\mathbf{B}}(f_{w_i}(a_i) \mid i \in |w|), s \\ &= F_{d(\sigma)}^{\mathbf{B}}(\widehat{f}_{\varphi(w_i)}^{\sharp}(a_i, w_i) \mid i \in |w|), s \\ &= \widehat{f}_t^{\sharp}(d(\sigma)((a_i, w_i) \mid i \in |w|)), \end{aligned}$$

therefore, $R^{\mathbf{A}} \subseteq \text{Ker}(\widehat{f}^{\sharp})$.

Since \mathbf{d}^* is a functor, $\mathbf{d}^*(f^\natural)$ is a Σ -homomorphism. Besides, for every $s \in S$ and $a \in A_s$, we have that

$$\begin{aligned} \mathbf{d}^*(f^\natural)_s(\eta_s^{\mathbf{A}}(a)) &= f_{\varphi(s)}^\natural([[a, s]]) \\ &= \widehat{f}_{\varphi(s)}^\natural(a, s) \\ &= \widehat{f}_{\varphi(s)}^\natural(\eta_{\prod_{\varphi} A, \varphi(s)}(a)) \\ &= \widehat{f}_{\varphi(s)}(a) \\ &= f_s(a), \end{aligned}$$

hence $f = \mathbf{d}^*(f^\natural) \circ \eta^{\mathbf{A}}$.

It is obvious that f^\natural is the unique \mathbf{A} -homomorphism from $\mathbf{d}_*(\mathbf{A})$ into \mathbf{B} such that the above diagram commutes, hence $\mathbf{d}_* \dashv \mathbf{d}^*$. \square

Proposition 12. *The category \mathbf{Alg} is cocomplete.*

Proof. The category \mathbf{Sig} is cocomplete. For every signature Σ , the category $\mathbf{Alg}(\Sigma)$ is cocomplete. The functor \mathbf{Alg} is locally reversible. Therefore, by Proposition 3, the category \mathbf{Alg} is cocomplete. \square

From Propositions 10 & 12 we obtain immediately the following

Corollary 2. *The category \mathbf{Alg} is bicomplete.*

The contravariant functor \mathbf{Alg} from \mathbf{Sig} to \mathbf{Cat} is not only useful to construct the category \mathbf{Alg} , actually, as we prove in what follows, it, together with a pseudo-functor \mathbf{Ter} from \mathbf{Sig} to \mathbf{Cat} , and a pseudo-extranatural transformation (\mathbf{Tr}, θ) (from a pseudo-functor on $\mathbf{Sig}^{\text{op}} \times \mathbf{Sig}$ to \mathbf{Cat} , induced by \mathbf{Alg} and \mathbf{Ter} , to the functor, between the same categories, constantly \mathbf{Set}), enables us to construct a new institution on \mathbf{Set} , the so-called many-sorted term institution, denoted by $\mathfrak{Tm} = (\mathbf{Sig}, \mathbf{Alg}, \mathbf{Ter}, (\mathbf{Tr}, \theta))$, but, we point out, for a concept of institution that is *strictly* more general than that of generalized \mathbf{V} -institution in [25]. The institution \mathfrak{Tm} can be qualified of basic, or fundamental, among others, by the following reasons:

- (1) It embodies, in a coherent way, algebras, terms, and the natural process of realization of terms as term operations in algebras, and
- (2) The many-sorted equational institution and the many-sorted specification institution (both to be defined in the third section), i.e., the core of universal algebra, are built on it.

It happens that, in the institution \mathfrak{Tm} on \mathbf{Set} , the existence of the pseudo-functor \mathbf{Ter} follows from the fact that, for any signature Σ , the terms for Σ , understood in a generalized sense to be explained below, have a category-theoretical interpretation as the morphisms of a category $\mathbf{Ter}(\Sigma)$. Furthermore, the component \mathbf{Tr} of the pseudo-extranatural transformation (\mathbf{Tr}, θ) , in \mathfrak{Tm} , depends for its existence on the fact that the generalized terms have associated generalized term operations on the algebras. And, finally, as it could not be otherwise, it is the case that the properties of the generalized terms, resp., generalized term operations on the algebras, are function of those of the ordinary terms, resp., term operations on the algebras.

Therefore, to proceed properly, we should begin by defining, for a Σ -algebra \mathbf{A} and an S -sorted set X , the concept of many-sorted X -ary operation on \mathbf{A} , that of many-sorted X -ary term operation on \mathbf{A} , and, as an immediate consequence of the universal property of the free algebras, the procedure of realization of terms P of type (X, s) as term operations $P^{\mathbf{A}}$ on \mathbf{A} (i.e., the passage from a formal operation P , constructed from variables and formal operations, to their realization as a true, or substantial, operation $P^{\mathbf{A}}$ on the algebra \mathbf{A} , that transforms valuations of the

variables X in the underlying ms-set A of \mathbf{A} , into elements, of the adequate sort, of \mathbf{A}).

Definition 5. Let X be an S -sorted set, \mathbf{A} a Σ -algebra, $s \in S$ and $P \in \mathbf{T}_\Sigma(X)_s$ a term for Σ of type (X, s) . Then

- (1) The Σ -algebra of the *many-sorted X -ary operations on \mathbf{A}* , $\mathbf{Op}_X(\mathbf{A})$, is \mathbf{A}^{A^X} , i.e., the direct A_X -power of \mathbf{A} , where A_X is $\text{Hom}(X, A)$, the (ordinary) set of the S -sorted mappings from X to A . From now on, to shorten terminology, we will speak of *X -ary operations on \mathbf{A}* instead of *many-sorted X -ary operations on \mathbf{A}* .
- (2) The Σ -algebra of the *many-sorted X -ary term operations on \mathbf{A}* , $\mathbf{Ter}_X(\mathbf{A})$, is the subalgebra of $\mathbf{Op}_X(\mathbf{A})$ generated by the subfamily

$$\mathcal{P}_X^A = (\mathcal{P}_{X,s}^A)_{s \in S} = (\{\text{pr}_{X,s,x}^A \mid x \in X_s\})_{s \in S}$$

of $\mathbf{Op}_X(A) = A^{A^X}$, where, for every $s \in S$ and $x \in X_s$, $\text{pr}_{X,s,x}^A$ is the mapping from A_X to A_s which sends $a \in A_X$ to $a_s(x)$. From now on, to shorten terminology, we will speak of *X -ary term operations on \mathbf{A}* instead of *many-sorted X -ary term operations on \mathbf{A}* .

- (3) We denote by $\text{Tr}^{X,\mathbf{A}}$ the unique Σ -homomorphism from $\mathbf{T}_\Sigma(X)$ to $\mathbf{Op}_X(\mathbf{A})$ such that $\text{pr}_X^A = \text{Tr}^{X,\mathbf{A}} \circ \eta_X$, where pr_X^A is the S -sorted mapping $(\text{pr}_{X,s}^A)_{s \in S}$ from X to $\mathbf{Op}_X(A)$, with $\text{pr}_{X,s}^A = (\text{pr}_{X,s,x}^A)_{x \in X_s}$, for every $s \in S$. Furthermore, $P^{\mathbf{A}}$ denotes the image of P under $\text{Tr}_s^{X,\mathbf{A}}$, and we call the mapping $P^{\mathbf{A}}$ from A_X to A_s , the *term operation on \mathbf{A} determined by P* , or the *term realization of P on \mathbf{A}* .

We recall that, for the insertion of generators $\eta_X: X \longrightarrow \mathbf{T}_\Sigma(X)$, $\text{Tr}^{X,\mathbf{A}}[\eta_X[X]]$, the (direct) image of $\eta_X[X]$ under $\text{Tr}^{X,\mathbf{A}}$, is also $\mathbf{Ter}_X(\mathbf{A})$, i.e., the term operations on an algebra are the same as the operations determined by the terms built from variables and formal operations denoting the primitive operations of the algebra. From now on, to simplify the notation, we will also denote by $\text{Tr}^{X,\mathbf{A}}$ the co-restriction of the Σ -homomorphism $\text{Tr}^{X,\mathbf{A}}$ from $\mathbf{T}_\Sigma(X)$ to $\mathbf{Op}_X(\mathbf{A})$ to the subalgebra $\mathbf{Ter}_X(\mathbf{A})$ of $\mathbf{Op}_X(\mathbf{A})$.

Remark. What we have called *term operations on \mathbf{A}* are also known, for those following the terminology in Grätzer [28], pp. 37–45, and Jónsson [37], pp. 83–87, as *polynomial operations of \mathbf{A}* , and, for those following that one in Cohn [14], pp. 145–149, as *derived operators of \mathbf{A}* .

Remark. If for an S -sorted set X , a Σ -algebra \mathbf{A} , and an S -sorted subset M of A , we define the Σ -algebra of the *X -ary term operations with constants in M on \mathbf{A}* , denoted by $\mathbf{Ter}_X(\mathbf{A}, M)$, as the subalgebra of $\mathbf{Op}_X(\mathbf{A})$ generated by the subfamily $\mathcal{P}_X^A \cup \mathcal{K}_X^{A,M}$ of $\mathbf{Op}_X(A) = A^{A^X}$, where

$$\mathcal{K}_X^{A,M} = (\mathcal{K}_{X,s}^{A,M})_{s \in S} = (\{\kappa_{X,s}^a \mid a \in M_s\})_{s \in S},$$

and, for every $s \in S$ and $a \in M_s$, $\kappa_{X,s}^a$ is the mapping from A_X to A_s that is constantly a , then we have that $\mathbf{Ter}_X(\mathbf{A})$ is $\mathbf{Ter}_X(\mathbf{A}, (\emptyset)_{s \in S})$, where $(\emptyset)_{s \in S}$ is the S -sorted set that is constantly \emptyset . And, using the terminology in Jónsson [37], pp. 87–89, the Σ -algebra of the *X -ary algebraic operations on \mathbf{A}* , denoted by $\mathbf{Alg}_X(\mathbf{A})$, is $\mathbf{Ter}_X(\mathbf{A}, A)$.

We point out that since the above concepts are defined for arbitrary ms-sets, they are also applicable, in particular, for a given set of sorts S and an arbitrary, but fixed, S -sorted set of variables $V^S = (V_s^S)_{s \in S}$, where, for every $s \in S$, $V_s^S = \{v_n^s \mid n \in \mathbb{N}\}$ is an effectively enumerated set, to the *finite S -sorted subsets $\downarrow w$ of V^S* associated to the words $w \in S^*$, where, for every word $w \in S^*$, we agree that

$\downarrow w$ is the finite subset of V^S defined, for every $s \in S$, as $(\downarrow w)_s = \{v_i^s \mid i \in w^{-1}[s]\}$ (observe that $(\downarrow w)_s$ is empty for those sorts s that do not occur in the word w).

In all that follows, every proposition relative to the above concepts will only be stated for arbitrary ms-sets, therefore the corresponding propositions for the finite S -sorted subsets $\downarrow w$ of V^S will not be actually stated and will remain tacit. However, to prove the just mentioned implicit propositions, it will be shown to be useful to know that, for a word $w \in S^*$, a mapping $\varphi: S \longrightarrow T$, and its extension $\varphi^*: \mathbf{T}_*(S) \longrightarrow \mathbf{T}_*(T)$ to the corresponding free monoids on S and T , the S -sorted set $\downarrow w$ can be embedded in the S -sorted set $(\downarrow \varphi^*(w))_\varphi$, associated to the T -sorted set $\downarrow \varphi^*(w) \cong \coprod_\varphi(\downarrow w)$, through the S -sorted mapping $\text{in}^{w,\varphi}$ defined, for every $s \in S$, as follows

$$\text{in}_s^{w,\varphi} \begin{cases} (\downarrow w)_s & \longrightarrow (\downarrow \varphi^*(w))_{\varphi(s)} \\ v_i^s & \longmapsto v_i^{\varphi(s)} \end{cases}$$

and that this embedding has as an immediate consequence that a signature morphism $\mathbf{d}: \Sigma \longrightarrow \Lambda$, determines a morphism

$$(\varphi, d, \text{in}^{w,\varphi}): (S, \Sigma, \downarrow w) \longrightarrow (T, \Lambda, \downarrow \varphi^*(w))$$

in $\mathbf{MSet} \times_{\mathbf{set}} \mathbf{Sig}$, hence that the s -th component of the Σ -homomorphism

$$(\text{in}^{w,\varphi})^{\mathbf{d}}: \mathbf{T}_\Sigma(\downarrow w) \longrightarrow \mathbf{T}_\Lambda(\downarrow \varphi^*(w))_\varphi$$

translates terms for Σ of type $(\downarrow w, s)$ into terms for Λ of type $(\downarrow \varphi^*(w), \varphi(s))$.

For completeness we recall that for many-sorted terms, as for single-sorted terms, we also have that

- (1) The *exchange law* is valid, i.e., that given a valuation $a: X \longrightarrow A$, where X is an S -sorted set and A the underlying set of a Σ -algebra \mathbf{A} , and a term P for Σ of type (X, s) , we always have the equality $a_s^\sharp(P) = P^{\mathbf{A}}(a)$; and that
- (2) The Σ -homomorphisms *commute* with term operations, i.e., that given a Σ -homomorphism $u: \mathbf{A} \longrightarrow \mathbf{B}$ and a term P for Σ of type (X, s) , we always have the equality $u_s \circ P^{\mathbf{A}} = P^{\mathbf{B}} \circ u_X$.

Following this we state the fundamental facts about term operations of different arities on the same algebra. These facts are, actually, the generalization to the many-sorted case and categorization of some of those stated by Schmidt in [54], pp. 107–109.

Proposition 13. *Let \mathbf{A} be a Σ -algebra and $f: X \longrightarrow Y$ an S -sorted mapping. Then there exists a unique Σ -homomorphism $\text{Ter}_f(\mathbf{A})$ from $\mathbf{Ter}_X(\mathbf{A})$ to $\mathbf{Ter}_Y(\mathbf{A})$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathbf{T}_\Sigma(X) & \xrightarrow{\text{Tr}^{X,\mathbf{A}}} & \mathbf{Ter}_X(\mathbf{A}) \\ f^\circledast \downarrow & & \downarrow \text{Ter}_f(\mathbf{A}) \\ \mathbf{T}_\Sigma(Y) & \xrightarrow{\text{Tr}^{Y,\mathbf{A}}} & \mathbf{Ter}_Y(\mathbf{A}) \end{array}$$

Besides, we have that

- (1) For every S -sorted set X , it is the case that

$$\text{Ter}_{\text{id}_X}(\mathbf{A}) = \text{id}_{\mathbf{Ter}_X(\mathbf{A})}.$$

- (2) If $g: Y \longrightarrow Z$ is another S -sorted mappings, then

$$\text{Ter}_{g \circ f}(\mathbf{A}) = \text{Ter}_g(\mathbf{A}) \circ \text{Ter}_f(\mathbf{A}).$$

Proof. As we know $\mathbf{Ter}_X(\mathbf{A})$ is the subalgebra of $\mathbf{Op}_X(\mathbf{A})$ generated by the subfamily \mathcal{P}_X^A of $\mathbf{Op}_X(A)$. Therefore, to prove that there is some Σ -homomorphism from $\mathbf{Ter}_X(\mathbf{A})$ to $\mathbf{Ter}_Y(\mathbf{A})$ it will be enough to prove that there is a Σ -homomorphism $\mathbf{Op}_f(\mathbf{A})$ from $\mathbf{Op}_X(\mathbf{A})$ to $\mathbf{Op}_Y(\mathbf{A})$ such that $\mathbf{Op}_f(\mathbf{A})[\mathcal{P}_X^A] \subseteq \mathcal{P}_Y^A$.

Let $\mathbf{Op}_f(\mathbf{A})$ be the S -sorted mapping from $\mathbf{Op}_X(A)$ to $\mathbf{Op}_Y(A)$ whose s -th coordinate mapping $\mathbf{Op}_f(\mathbf{A})_s$, for $s \in S$, sends a mapping P in $A_s^{A_X}$ to the mapping $P \circ A_f$ in $A_s^{A_Y}$, where A_f is the mapping from A_Y to A_X which assigns to an S -sorted mapping u in A_Y the S -sorted mapping $u \circ f$ in A_X . Thus defined $\mathbf{Op}_f(\mathbf{A})$ is a Σ -homomorphism from $\mathbf{Op}_X(\mathbf{A})$ to $\mathbf{Op}_Y(\mathbf{A})$.

Furthermore, for every $s \in S$, the action of $\mathbf{Op}_f(\mathbf{A})_s$ on a generator $\text{pr}_{X,s,x}^A$ of $\mathbf{Ter}_X(\mathbf{A})$ is the mapping $\mathbf{Op}_f(\mathbf{A})_s(\text{pr}_{X,s,x}^A) = \text{pr}_{X,s,x}^A \circ A_f$ from A_Y to A_s . But for an S -sorted mapping u in A_Y , we have that $(\mathbf{Op}_f(\mathbf{A})_s(\text{pr}_{X,s,x}^A))(u) = u_s(f_s(x))$, therefore $\mathbf{Op}_f(\mathbf{A})_s(\text{pr}_{X,s,x}^A) = \text{pr}_{Y,s,f_s(x)}^A$, that is a member of $\mathcal{P}_{Y,s}^A$, the s -th component of the set of generators \mathcal{P}_Y^A of $\mathbf{Ter}_Y(\mathbf{A})$. From this we can assert that $\mathbf{Op}_f(\mathbf{A})$ has a bi-restriction, $\mathbf{Ter}_f(\mathbf{A})$, to $\mathbf{Ter}_X(\mathbf{A})$ and $\mathbf{Ter}_Y(\mathbf{A})$. Thus defined $\mathbf{Ter}_f(\mathbf{A})$ is the unique Σ -homomorphism from $\mathbf{Ter}_X(\mathbf{A})$ to $\mathbf{Ter}_Y(\mathbf{A})$ such that $\mathbf{Ter}_f(\mathbf{A}) \circ \text{Tr}^{X,\mathbf{A}} = \text{Tr}^{Y,\mathbf{A}} \circ f^\circledast$.

The remaining properties follow easily from the definition of $\mathbf{Ter}_f(\mathbf{A})$. \square

The proposition just stated can be interpreted as meaning that, for a Σ -algebra \mathbf{A} , we have

- (1) A functor $\text{Ter}_{(\cdot)}(\mathbf{A})$ from \mathbf{Set}^S to $\mathbf{Alg}(\Sigma)$ which sends an S -sorted set X to the Σ -algebra $\mathbf{Ter}_X(\mathbf{A})$, and an S -sorted mapping f from X to Y to the Σ -homomorphism $\mathbf{Ter}_f(\mathbf{A})$ from $\mathbf{Ter}_X(\mathbf{A})$ to $\mathbf{Ter}_Y(\mathbf{A})$, and
- (2) A natural transformation $\text{Tr}^{(\cdot),\mathbf{A}}$ from \mathbf{T}_Σ to $\text{Ter}_{(\cdot)}(\mathbf{A})$ which sends an S -sorted set X to the Σ -homomorphism $\text{Tr}^{X,\mathbf{A}}$ from $\mathbf{T}_\Sigma(X)$ to $\mathbf{Ter}_X(\mathbf{A})$,

summarized in the following diagram

$$\begin{array}{ccc}
 & \mathbf{T}_\Sigma & \\
 \text{Set}^S & \begin{array}{c} \curvearrowright \\ \Downarrow \text{Tr}^{(\cdot),\mathbf{A}} \\ \curvearrowleft \end{array} & \mathbf{Alg}(\Sigma) \\
 & \text{Ter}_{(\cdot)}(\mathbf{A}) &
 \end{array}$$

What we want to prove now is the compatibility between the translation of terms and their realization as term operations on the algebras. But for this it will be shown to be useful to take into account the following auxiliary functors and natural transformation.

Definition 6. For a mapping $\varphi: S \longrightarrow T$, an S -sorted set X , a T -sorted set Y , and an S -sorted mapping $f: X \longrightarrow Y_\varphi$, we have the following functors and natural transformation

- (1) $\mathbf{H}(Y, \cdot)$ is the covariant hom-functor from \mathbf{Set}^T to \mathbf{Set} which, we recall, sends a T -sorted set A to the set $\mathbf{H}(Y, \cdot)(A) = A_Y$, and a T -sorted mapping u from A to B to the mapping $\mathbf{H}(Y, \cdot)(u)$ from A_Y to B_Y which assigns to a T -sorted mapping t from Y to A the mapping $u \circ t$ from Y to B .
- (2) $\mathbf{H}(X, \cdot) \circ \Delta_\varphi$ is the functor from \mathbf{Set}^T to \mathbf{Set} which sends a T -sorted set A to the set $(A_\varphi)_X$, and a T -sorted mapping u from A to B to the mapping $\mathbf{H}(X, \cdot)(u_\varphi)$ from $(A_\varphi)_X$ to $(B_\varphi)_X$ which assigns to an S -sorted mapping ℓ from X to A_φ the mapping $u_\varphi \circ \ell$ from X to B_φ .

- (3) $\vartheta^{\varphi, f}$ is the natural transformation from $H(Y, \cdot)$ to $H(X, \cdot) \circ \Delta_\varphi$, as in the following diagram

$$\begin{array}{ccc} & H(Y, \cdot) & \\ \text{Set}^T \curvearrowright & \Downarrow \vartheta^{\varphi, f} & \curvearrowleft \text{Set} \\ & H(X, \cdot) \circ \Delta_\varphi & \end{array}$$

which sends a T -sorted set A to the mapping $\vartheta_A^{\varphi, f}$ from A_Y to $(A_\varphi)_X$ which assigns to a morphism $t: Y \rightarrow A$ in A_Y the morphism $t_\varphi \circ f$ in $(A_\varphi)_X$.

From this definition, for a T -sorted set A , we get the S -sorted mapping $\Upsilon_A^{\varphi, f}$ from $\text{Op}_X(A_\varphi) = A_\varphi^{(A_\varphi)_X}$ to $\text{Op}_Y(A)_\varphi = (A^{A_Y})_\varphi = A_\varphi^{A_Y}$ which sends, for $s \in S$, a mapping $a: (A_\varphi)_X \rightarrow A_{\varphi(s)}$ to the mapping $a \circ \vartheta_A^{\varphi, f}: A_Y \rightarrow A_{\varphi(s)}$, that we will use in the proof of the following proposition and corollary.

Proposition 14. *Let $(\varphi, d, f): (S, \Sigma, X) \rightarrow (T, \Lambda, Y)$ be a morphism in the category $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$. Then, for every Λ -algebra \mathbf{A} and term $P \in \mathbf{T}_\Sigma(X)_s$ for Σ of type (X, s) , the following diagram commutes*

$$\begin{array}{ccc} (A_\varphi)_X & \xrightarrow{P^{\mathbf{d}^*(\mathbf{A})}} & A_{\varphi(s)} \\ \vartheta_A^{\varphi, f} \uparrow & & \parallel \\ A_Y & \xrightarrow{f_s^{\mathbf{d}}(P)^{\mathbf{A}}} & A_{\varphi(s)} \end{array}$$

Proof. Let $a \in A_Y$ be a T -sorted mapping from Y to A . Then the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathbf{T}_\Sigma(X) \\ f \downarrow & & \downarrow f^{\mathbf{d}} \\ Y_\varphi & \xrightarrow{(\eta_Y)_\varphi} & \mathbf{T}_\Lambda(Y)_\varphi \\ & \searrow a_\varphi & \downarrow (a^\#)_\varphi \\ & & A_\varphi \end{array} \quad \left. \begin{array}{l} \text{---} \\ \text{---} \\ \text{---} \end{array} \right\} (a_\varphi \circ f)^\#$$

hence, for every $P \in \mathbf{T}_\Sigma(X)_s$, we have that

$$\begin{aligned} f_s^{\mathbf{d}}(P)^{\mathbf{A}}(a) &= (a^\#)_{\varphi(s)} \circ f_s^{\mathbf{d}}(P) \\ &= (a_\varphi \circ f)^\#(P) \\ &= P^{\mathbf{d}^*(\mathbf{A})}(a_\varphi \circ f) \\ &= P^{\mathbf{d}^*(\mathbf{A})} \circ \vartheta_A^{\varphi, f}(a). \end{aligned}$$

Therefore $f_s^{\mathbf{d}}(P)^{\mathbf{A}} = P^{\mathbf{d}^*(\mathbf{A})} \circ \vartheta_A^{\varphi, f}$, as asserted. \square

We gather in the following corollary some useful consequences of the last proposition.

Corollary 3. *Let $(\varphi, d, f): (S, \Sigma, X) \rightarrow (T, \Lambda, Y)$ be a morphism in the category $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, \mathbf{A} a Λ -algebra, and $P \in \mathbf{T}_\Sigma(X)_s$ a term for Σ of type (X, s) . Then we have that*

(1) *The following diagrams commute*

$$\begin{array}{ccc}
\mathbf{T}_\Sigma(X) & \xrightarrow{\mathrm{Tr}^{X, \mathbf{d}^*(\mathbf{A})}} & \mathbf{Ter}_X(\mathbf{d}^*(\mathbf{A})) \\
f^{\mathbf{d}} \downarrow & & \downarrow \Upsilon_A^{\varphi, f} \\
\mathbf{T}_\Lambda(Y)_\varphi & \xrightarrow{\mathrm{Tr}_\varphi^{Y, \mathbf{A}}} & \mathbf{Ter}_Y(\mathbf{A})_\varphi
\end{array}
\qquad
\begin{array}{ccc}
(A_\varphi)_X & \xrightarrow{P^{\mathbf{d}^*(\mathbf{A})}} & A_{\varphi(s)} \\
\theta_{X, \mathbf{A}}^\varphi \uparrow & & \parallel \\
A_{\coprod_\varphi X} & \xrightarrow{\eta_{X, s}^{\mathbf{d}^*(\mathbf{A})}} & A_{\varphi(s)}
\end{array}$$

(2) *If $(\varphi, d, g): (S, \Sigma, X') \longrightarrow (T, \Lambda, Y')$ is another morphism in the category $\mathbf{MSet} \times_{\mathbf{Set}} \mathbf{Sig}$, k an S -sorted mapping from X to X' , and ℓ a T -sorted mapping from Y to Y' such that $\ell_\varphi \circ f = g \circ k$, then the following diagram commutes*

$$\begin{array}{ccccc}
& & \mathbf{T}_\Sigma(X') & \xrightarrow{\mathrm{Tr}^{X', \mathbf{d}^*(\mathbf{A})}} & \mathbf{Ter}_{X'}(\mathbf{d}^*(\mathbf{A})) \\
& & \downarrow g^{\mathbf{d}} & & \downarrow \mathrm{Ter}_k(\mathbf{d}^*(\mathbf{A})) \\
\mathbf{T}_\Sigma(X) & \xrightarrow{k^\circledast} & \mathbf{T}_\Sigma(X') & \xrightarrow{\mathrm{Tr}^{X, \mathbf{d}^*(\mathbf{A})}} & \mathbf{Ter}_X(\mathbf{d}^*(\mathbf{A})) \\
\downarrow f^{\mathbf{d}} & & \downarrow g^{\mathbf{d}} & & \downarrow \Upsilon_A^{\varphi, f} \\
\mathbf{T}_\Lambda(Y)_\varphi & \xrightarrow{\ell^\circledast} & \mathbf{T}_\Lambda(Y')_\varphi & \xrightarrow{\mathrm{Tr}_\varphi^{Y', \mathbf{A}}} & \mathbf{Ter}_{Y'}(\mathbf{A})_\varphi \\
& & \downarrow \Upsilon_A^{\varphi, f} & & \downarrow \Upsilon_A^{\varphi, g} \\
& & \mathbf{Ter}_Y(\mathbf{A})_\varphi & \xrightarrow{\mathrm{Ter}_\ell(\mathbf{A})_\varphi} & \mathbf{Ter}_Y(\mathbf{A})_\varphi
\end{array}$$

Proof. We restrict ourselves to prove the first part of the corollary. The upper left-hand diagram commutes because, for a morphism (φ, d, f) from (S, Σ, X) to (T, Λ, Y) and a Λ -algebra \mathbf{A} , the S -sorted mapping $\Upsilon_A^{\varphi, f}$ from $\mathrm{Op}_X(A_\varphi)$ to $\mathrm{Op}_Y(A)_\varphi$ is actually a Σ -homomorphism from $\mathbf{Op}_X(\mathbf{d}^*(\mathbf{A}))$ to $\mathbf{Op}_Y(\mathbf{A})_\varphi$ that restricts to $\mathbf{Ter}_X(\mathbf{d}^*(\mathbf{A}))$ and $\mathbf{Ter}_Y(\mathbf{A})_\varphi$.

The upper right-hand diagram commutes because, for the T -sorted set $\coprod_\varphi X$ and the S -sorted mapping η_X^φ from X to $(\coprod_\varphi X)_\varphi$, we have that $\vartheta_A^{\varphi, \eta_X^\varphi} = \theta_{X, \mathbf{A}}^\varphi$. \square

As is well-known, for a signature Σ , the conglomerate of terms for Σ is precisely the set $\bigcup_{X \in \mathbf{U}} \bigcup_{s \in S} \mathbf{T}_\Sigma(X)_s$, but such an amorphous set is not adequate, because of its lack of structure, for some tasks, as e.g., to explain the invariant character of the realization of terms as term operations on algebras, under change of signature (or to state a Completeness Theorem for finitary many-sorted equational logic).

However, by conveniently generalizing the concept of term for a signature Σ (as explained immediately below), it is possible to endow, in a natural way, to the corresponding generalized terms for Σ , taken as *morphisms*, with a structure of category, that enables us to give, in this paper, a category-theoretical explanation of the existing relation between terms and algebras. To this we add that the use of the generalized terms and related notions, such as, e.g., that of generalized equation (to be defined in the following section), has allowed us, in [13], to provide a purely category-theoretical proof of the Completeness Theorem for monads in categories of sorted sets. Moreover, such a proof, after dualizing the generalized terms and equations, is also applicable to get a corresponding Completeness Theorem for comonads in categories of sorted sets.

Actually, we associate to every signature Σ the category $\mathbf{Kl}(\mathbf{T}_\Sigma)^{\mathrm{op}}$, of generalized terms for Σ , that we denote, to shorten notation, by $\mathbf{Ter}(\Sigma)$, i.e., the dual of the Kleisli category for $\mathbf{T}_\Sigma = (\mathbf{T}_\Sigma, \eta, \mu)$, the standard monad derived from the adjunction $\mathbf{T}_\Sigma \dashv \mathbf{G}_\Sigma$ between the category $\mathbf{Alg}(\Sigma)$ and the category \mathbf{Set}^S , with

$\mathbb{T}_\Sigma = \mathbb{G}_\Sigma \circ \mathbb{T}_\Sigma$. Thus we will be working with a category, $\mathbf{Ter}(\Sigma)$ (to be defined fully below), that has as objects the S -sorted sets and as morphisms from an S -sorted set X into a like one Y the S -sorted mappings P from Y to $\mathbb{T}_\Sigma(X)$, i.e., the families $P = (P_s)_{s \in S}$, where, for every $s \in S$, P_s is a mapping from Y_s to $\mathbb{T}_\Sigma(X)_s$, which sends a variable $y \in Y_s$ to the term $P_s(y) \in \mathbb{T}_\Sigma(X)_s$. These morphisms in $\mathbf{Ter}(\Sigma)$ from X to Y we will call generalized terms for Σ of type (X, Y) , or, simply, terms for Σ of type (X, Y) .

The construction of the category $\mathbf{Ter}(\Sigma)$ is a natural one. This is so, essentially, because it has been obtained by applying a category-theoretical construction, concretely that of Kleisli (in [38]). However, to understand more plainly how the category $\mathbf{Ter}(\Sigma)$ is obtained, or, more precisely, from where the morphisms of $\mathbf{Ter}(\Sigma)$ arise, the following observation could be helpful. For a signature Σ , an S -sorted set X , and a sort $s \in S$, an ordinary term $P \in \mathbb{T}_\Sigma(X)_s$ for Σ of type (X, s) is, essentially, an S -sorted mapping $P: \delta^s \longrightarrow \mathbb{T}_\Sigma(X)$ where, for $s \in S$, $\delta^s = (\delta_t^s)_{t \in S}$, the delta of Kronecker in s , is the S -sorted set such that $\delta_t^s = \emptyset$ if $s \neq t$ and $\delta_s^s = 1$. But the just mentioned S -sorted mappings do not constitute the morphisms of a category. Therefore, in order to get a category, it seems natural to replace the special S -sorted sets that are the deltas of Kronecker, as domains of morphisms, by arbitrary S -sorted sets, thus obtaining the generalized terms, that are the category-theoretical rendering of the ordinary terms, since they are now S -sorted mappings from an S -sorted set to the free Σ -algebra on another S -sorted set, i.e., morphisms in a category $\mathbf{Ter}(\Sigma)$.

This category-theoretical perspective about terms, in its turn, will allow us to get a functor Tr^Σ , of realization of terms as term operations, from $\mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Sigma)$ to \mathbf{Set} , and therefore to define (in the next section) the validation of equations, understood as ordered pairs of coterminial terms in the corresponding generalized sense, in an algebra.

Since it will be fundamental in all that follows, we provide, for a signature Σ , the full definition of the category $\mathbf{Ter}(\Sigma)$, as announced above, and also the definition of the procedure of realization of the terms for Σ as term operations on a given Σ -algebra. Observe that we depart, in the definition of the category $\mathbf{Ter}(\Sigma)$, but only for this type of category, from the (non-Ehresmannian) tradition, in calling a category by the name of its morphisms.

Definition 7. Let Σ be a signature and \mathbf{A} a Σ -algebra. Then

- (1) The category of *terms for Σ* , $\mathbf{Ter}(\Sigma)$, is the dual of $\mathbf{Kl}(\mathbb{T}_\Sigma)$. Therefore $\mathbf{Ter}(\Sigma)$ has
 - (a) As objects the elements of \mathcal{U}^S , i.e., the S -sorted sets,
 - (b) As morphisms from X to Y , that we call *terms for Σ of type (X, Y)* , or, simply, *terms of type (X, Y)* , the S -sorted mappings from Y to $\mathbb{T}_\Sigma(X)$,
 - (c) As composition, denoted in $\mathbf{Ter}(\Sigma)$ and $\mathbf{Kl}(\mathbb{T}_\Sigma)$ by \diamond , the operation which sends terms $P: X \longrightarrow Y$ and $Q: Y \longrightarrow Z$ in $\mathbf{Ter}(\Sigma)$ to the term $Q \diamond P: X \longrightarrow Z$ in $\mathbf{Ter}(\Sigma)$ defined as

$$Q \diamond P = \mu_X \circ P^\circledast \circ Q,$$

where μ_X is the value at X of the multiplication μ of the monad $\mathbb{T}_\Sigma = (\mathbb{T}_\Sigma, \eta, \mu)$ and P^\circledast the value of the functor \mathbb{T}_Σ at the S -sorted mapping $P: Y \longrightarrow \mathbb{T}_\Sigma(X)$, and

- (d) As identities the values of η , the unit of the monad \mathbb{T}_Σ , in the S -sorted sets.
- (2) If $P: X \longrightarrow Y$ is a term for Σ of type (X, Y) , then $P^\mathbf{A}$, the *term operation on \mathbf{A} determined by P* , or the *term realization of P on \mathbf{A}* , is the

mapping from A_X to A_Y which assigns to a valuation f of the variables X in A the valuation $f^\# \circ P$ of the variables Y in A , i.e., the composition of $P: Y \longrightarrow \mathbf{T}_\Sigma(X)$ and the underlying mapping of $f^\#: \mathbf{T}_\Sigma(X) \longrightarrow \mathbf{A}$, the canonical extension of the valuation $f: X \longrightarrow A$.

Remark. For a term $P: X \longrightarrow Y$ for Σ of type (X, Y) , the term operation $P^{\mathbf{A}}$ on \mathbf{A} determined by P is also the mapping from A_X to A_Y obtained from the family

$$((\text{Tr}_s^{X, \mathbf{A}}(P_s(y)))_{y \in Y_s})_{s \in S} \in \prod_{s \in S} \text{Hom}(Y_s, \text{Hom}(A_X, A_s)),$$

through the following natural isomorphisms:

$$\begin{aligned} \prod_{s \in S} \text{Hom}(Y_s, \text{Hom}(A_X, A_s)) &\cong \prod_{s \in S} \text{Hom}(A_X, \text{Hom}(Y_s, A_s)) \\ &\cong \text{Hom}(A_X, \prod_{s \in S} \text{Hom}(Y_s, A_s)) \\ &\cong \text{Hom}(A_X, A_Y). \end{aligned}$$

After associating to every signature Σ the corresponding category $\mathbf{Ter}(\Sigma)$ of terms, we proceed to assign to every signature morphism $\mathbf{d}: \Sigma \longrightarrow \Lambda$, where, we recall, \mathbf{d} is a pair (φ, d) with φ a mapping from S to T and d a morphism in $\mathbf{Set}(S)$ from Σ to $\Lambda_{\varphi^* \times \varphi}$, a corresponding functor \mathbf{d}_\diamond from $\mathbf{Ter}(\Sigma)$ to $\mathbf{Ter}(\Lambda)$.

Proposition 15. *Let $\mathbf{d}: \Sigma \longrightarrow \Lambda$ be a signature morphism. Then there exists a functor \mathbf{d}_\diamond from $\mathbf{Ter}(\Sigma)$ to $\mathbf{Ter}(\Lambda)$ defined as follows*

- (1) \mathbf{d}_\diamond sends an S -sorted set X to the T -sorted set $\mathbf{d}_\diamond(X) = \prod_{\varphi} X$.
- (2) \mathbf{d}_\diamond sends a morphism P from X to Y in $\mathbf{Ter}(\Sigma)$ to the morphism $\mathbf{d}_\diamond(P) = (\theta^\varphi)^{-1}(\eta_X^{\mathbf{d}} \circ P)$ from $\prod_{\varphi} X$ to $\prod_{\varphi} Y$ in $\mathbf{Ter}(\Lambda)$, where $\eta_X^{\mathbf{d}}$ is the Σ -homomorphism from $\mathbf{T}_\Sigma(X)$ to $\mathbf{T}_\Lambda(\prod_{\varphi} X)_\varphi$ that extends the S -sorted mapping $(\eta_{\prod_{\varphi} X})_\varphi \circ \eta_X^\varphi$ from X to $\mathbf{T}_\Lambda(\prod_{\varphi} X)_\varphi$, i.e., for $\eta_X^{\mathbf{d}}$ we have that

$$\eta_X^{\mathbf{d}} = ((\eta_{\prod_{\varphi} X})_\varphi \circ \eta_X^\varphi)^\#,$$

θ^φ the natural isomorphism of the adjunction $\prod_{\varphi} \dashv \Delta_\varphi$, and η^φ the unit of the same adjunction.

Proof. To begin with, we prove that \mathbf{d}_\diamond preserves identities. If X is an S -sorted set, then the following diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathbf{T}_\Sigma(X) \\ \eta_X^\varphi \downarrow & \searrow \theta^\varphi(\eta_{\prod_{\varphi} X}) & \downarrow \eta_X^{\mathbf{d}} = (\theta^\varphi(\eta_{\prod_{\varphi} X}))^\# \\ (\prod_{\varphi} X)_\varphi & \xrightarrow{(\eta_{\prod_{\varphi} X})_\varphi} & \mathbf{T}_\Lambda(\prod_{\varphi} X)_\varphi \end{array}$$

commutes, therefore $\mathbf{d}_\diamond(\eta_X) = \eta_{\prod_{\varphi} X}$. Now we prove that \mathbf{d}_\diamond preserves compositions. Let $P: X \longrightarrow Y$ and $Q: Y \longrightarrow Z$ be morphisms in $\mathbf{Ter}(\Sigma)$. Then we have the following equations:

$$\begin{aligned} \mathbf{d}_\diamond(Q \diamond P) &= (\theta^\varphi)^{-1}(\eta_X^{\mathbf{d}} \circ P^\# \circ Q) \\ &= (\theta^\varphi)^{-1}(\eta_X^{\mathbf{d}}) \circ \prod_{\varphi} P^\# \circ \prod_{\varphi} Q, \\ \mathbf{d}_\diamond(Q) \diamond \mathbf{d}_\diamond(P) &= \mathbf{d}_\diamond(P)^\# \circ \mathbf{d}_\diamond(Q) \\ &= \mathbf{d}_\diamond(P)^\# \circ (\theta^\varphi)^{-1}(\eta_Y^{\mathbf{d}}) \circ \prod_{\varphi} Q, \end{aligned}$$

therefore, to prove that $\mathbf{d}_\circ(Q \diamond P) = \mathbf{d}_\circ(Q) \diamond \mathbf{d}_\circ(P)$ it is enough to verify the following equation

$$(\theta^\varphi)^{-1}(\eta_X^{\mathbf{d}}) \circ \coprod_\varphi P^\# = \mathbf{d}_\circ(P)^\# \circ (\theta^\varphi)^{-1}(\eta_Y^{\mathbf{d}}).$$

But for this, because of the commutativity of the following diagram

$$\begin{array}{ccccc}
 & & & & (\theta^\varphi)^{-1}(\eta_Y^{\mathbf{d}}) \\
 & & & & \downarrow \\
 \coprod_\varphi T_\Sigma(Y) & \xrightarrow{\coprod_\varphi \eta_Y^{\mathbf{d}}} & \coprod_\varphi T_\Lambda(\coprod_\varphi Y)_\varphi & \xrightarrow{\varepsilon_{T_\Lambda(\coprod_\varphi Y)}^\varphi} & T_\Lambda(\coprod_\varphi Y) \\
 \downarrow \coprod_\varphi P^\# & & \downarrow \coprod_\varphi \mathbf{d}_\circ(P)^\#_\varphi & & \downarrow \mathbf{d}_\circ(P)^\# \\
 \coprod_\varphi T_\Lambda(X) & \xrightarrow{\coprod_\varphi \eta_X^{\mathbf{d}}} & \coprod_\varphi T_\Lambda(\coprod_\varphi X)_\varphi & \xrightarrow{\varepsilon_{T_\Lambda(\coprod_\varphi X)}^\varphi} & T_\Lambda(\coprod_\varphi X) \\
 & & & & \uparrow (\theta^\varphi)^{-1}(\eta_X^{\mathbf{d}})
 \end{array}$$

it is enough to verify the following equation

$$(1) \quad \eta_X^{\mathbf{d}} \circ P^\# = \mathbf{d}_\circ(P)^\# \circ \eta_Y^{\mathbf{d}}.$$

But equation (1) is valid because the restriction of both terms to the generating ms-set Y coincide:

$$\begin{aligned}
 \eta_X^{\mathbf{d}} \circ P^\# \circ \eta_Y &= \eta_X^{\mathbf{d}} \circ P, \\
 \mathbf{d}_\circ(P)^\# \circ \eta_Y^{\mathbf{d}} \circ \eta_Y &= \mathbf{d}_\circ(P)^\# \circ (\eta_{\coprod_\varphi Y})_\varphi \circ \eta_Y^\varphi \\
 &= \mathbf{d}_\circ(P)_\varphi \circ \eta_Y^\varphi \\
 &= (\theta^\varphi)^{-1}(\eta_X^{\mathbf{d}} \circ P)_\varphi \circ \eta_Y^\varphi \\
 &= (\eta_X^{\mathbf{d}} \circ P)^\# \circ \eta_Y \\
 &= \eta_X^{\mathbf{d}} \circ P. \quad \square
 \end{aligned}$$

Remark. For a term $P: X \longrightarrow Y$, the term $\mathbf{d}_\circ(P): \coprod_\varphi X \longrightarrow \coprod_\varphi Y$ can be defined alternative, but equivalently, as the composition of the morphisms in the following diagram

$$\coprod_\varphi Y \xrightarrow{\coprod_\varphi P} \coprod_\varphi T_\Sigma(X) \xrightarrow{\coprod_\varphi \eta_X^{\mathbf{d}}} \coprod_\varphi T_\Lambda(\coprod_\varphi X)_\varphi \xrightarrow{\varepsilon_{T_\Lambda(\coprod_\varphi X)}^\varphi} T_\Lambda(\coprod_\varphi X),$$

where, recalling that $\eta^{\mathbf{d}} = \mathbf{G}_\Sigma * \eta^{\mathbf{d}}$ is the second natural transformation in Proposition 9 and ε^φ the counit of the adjunction $\coprod_\varphi \dashv \Delta_\varphi$, we have that

- (1) The T -sorted mapping $\coprod_\varphi \eta_X^{\mathbf{d}}$ is the component at X of the natural transformation $\coprod_\varphi * \eta^{\mathbf{d}}$ from $\coprod_\varphi \circ T_\Sigma$ to $\coprod_\varphi \circ \Delta_\varphi \circ T_\Lambda \circ \coprod_\varphi$, and
- (2) The T -sorted mapping $\varepsilon_{T_\Lambda(\coprod_\varphi X)}^\varphi$ is the component at X of the natural transformation $\varepsilon^\varphi * (T_\Lambda \circ \coprod_\varphi X)$ from $\coprod_\varphi \circ \Delta_\varphi \circ T_\Lambda \circ \coprod_\varphi$ to $T_\Lambda \circ \coprod_\varphi X$.

We state now for the generalized terms the homologous of the right-hand diagram in the first part of Corollary 3, i.e., the invariant character under signature change of the realization of terms as term operations in arbitrary, but fixed, algebras. We remark that from this fact we will get, in the third section, the invariance of the relation of satisfaction under signature change.

Proposition 16. *Let $\mathbf{d}: \Sigma \longrightarrow \Lambda$ be a signature morphism. Then, for every Λ -algebra \mathbf{A} and term $P: X \longrightarrow Y$ for Σ of type (X, Y) , the following diagram commutes*

$$\begin{array}{ccc} (A_\varphi)_X & \xrightarrow{P^{\mathbf{d}^*(\mathbf{A})}} & (A_\varphi)_Y \\ \theta_{X,A}^\varphi \uparrow & & \uparrow \theta_{Y,A}^\varphi \\ A_{\prod_\varphi X} & \xrightarrow{\mathbf{d}_\circ(P)^\mathbf{A}} & A_{\prod_\varphi Y} \end{array}$$

Proof. Because the S -sorted set Y is isomorphic to $\prod_{s \in S, y \in Y_s} \delta^s$ and the functor \prod_φ preserves colimits, since it has Δ_φ as a right adjoint, $\prod_\varphi Y$ is isomorphic to $\prod_{s \in S, y \in Y_s} \delta^{\varphi(s)}$. But $\text{Hom}(\prod_\varphi Y, A)$ and $\prod_{s \in S, y \in Y_s} \text{Hom}(\delta^{\varphi(s)}, A)$ are isomorphic, therefore it is enough to prove the proposition for the S -sorted sets of the type δ^s , i.e., the deltas of Kronecker, and this follows directly from Corollary 3. \square

Once defined the mappings that associate, respectively, to a signature the corresponding category of terms, and to a signature morphism the functor between the associated categories of terms, we state in the following proposition that both mappings are actually the components of a pseudo-functor from **Sig** to the 2-category **Cat**.

Proposition 17. *There exists a pseudo-functor Ter from **Sig** to the 2-category **Cat** given by the following data*

- (1) *The object mapping of Ter is that which sends a signature Σ to the category $\text{Ter}(\Sigma) = \mathbf{Ter}(\Sigma)$.*
- (2) *The morphism mapping of Ter is that which sends a signature morphism \mathbf{d} from Σ to Λ to the functor $\text{Ter}(\mathbf{d}) = \mathbf{d}_\circ$ from $\mathbf{Ter}(\Sigma)$ to $\mathbf{Ter}(\Lambda)$.*
- (3) *For every $\mathbf{d}: \Sigma \longrightarrow \Lambda$ and $\mathbf{e}: \Lambda \longrightarrow \Omega$, the natural isomorphism $\gamma^{\mathbf{d}, \mathbf{e}}$ from $\mathbf{e}_\circ \circ \mathbf{d}_\circ$ to $(\mathbf{e} \circ \mathbf{d})_\circ$ is that which is defined, for every S -sorted set X , as the isomorphism $\gamma_X^{\mathbf{d}, \mathbf{e}}: \prod_\psi \prod_\varphi X \longrightarrow \prod_{\psi \circ \varphi} X$ in $\mathbf{Ter}(\Omega)$ that corresponds to the U -sorted mapping*

$$\prod_{\psi \circ \varphi} X \xrightarrow{(\gamma_X^{\varphi, \psi})^{-1}} \prod_\psi \prod_\varphi X \xrightarrow{\eta_{\prod_\psi \prod_\varphi X}} \mathbf{T}_\Omega(\prod_\psi \prod_\varphi X),$$

where $\gamma_X^{\varphi, \psi}$ is the component at X of the natural isomorphism $\gamma^{\varphi, \psi}$ for the pseudo-functor MSet^{II} .

- (4) *For every signature Σ , the natural isomorphism ν^Σ from $\text{Id}_{\mathbf{Ter}(\Sigma)}$ to $(\text{id}_\Sigma)_\circ$ is that which is defined, for every S -sorted set X , as the isomorphism $\nu_X^\Sigma: X \longrightarrow \prod_{\text{id}_S} X$ in $\mathbf{Ter}(\Sigma)$ that corresponds to the S -sorted mapping*

$$\prod_{\text{id}_S} X \xrightarrow{\nu_X^S} X \xrightarrow{\eta_X} \mathbf{T}_\Omega(X),$$

where ν_X^S is the component at X of the natural isomorphism ν^S for the pseudo-functor MSet^{II} .

Our next goals are to prove that

- (1) For a signature Σ , there exists a functor Tr^Σ from the product category $\mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Sigma)$ to **Set**, that formalizes simultaneously the procedure of realization of terms (as term operations on algebras), and its naturalness (by taking into account the variation of the algebras through the homomorphisms between them), and that

- (2) For a signature morphism $\mathbf{d}: \Sigma \longrightarrow \Lambda$, there exists a natural isomorphism $\theta^{\mathbf{d}}$ from the functor $\mathrm{Tr}^{\Lambda} \circ (\mathrm{Id}_{\mathbf{Alg}(\Lambda)} \times \mathbf{d}_{\circ})$ to the functor $\mathrm{Tr}^{\Sigma} \circ (\mathbf{d}^* \times \mathrm{Id}_{\mathbf{Ter}(\Sigma)})$, that shows the invariant character of the procedure of realization of terms under signature change.

To accomplish the first stated goal we begin by proving the following

Lemma 1. *Let \mathbf{A} be a Σ -algebra, P a term of type (X, Y) , and Q a term of type (Y, Z) . Then we have that*

- (1) $(Q \diamond P)^{\mathbf{A}} = Q^{\mathbf{A}} \circ P^{\mathbf{A}}$.
- (2) For η_X , the identity morphism at X in $\mathbf{Ter}(\Sigma)$, $\eta_X^{\mathbf{A}} = \mathrm{id}_{A_X}$.

Proof. We restrict ourselves to prove the first part of the lemma. Since $(Q \diamond P)^{\mathbf{A}}$ is the mapping from A_X to A_Z which sends an S -sorted mapping $u: X \longrightarrow A$ to the S -sorted mapping

$$u^{\sharp} \circ (Q \diamond P) = u^{\sharp} \circ \mu_X \circ P^{\circledast} \circ Q: Z \longrightarrow A,$$

where, we recall, μ_X is the value in X of the multiplication μ of the monad $\mathbb{T}_{\Sigma} = (\mathbb{T}_{\Sigma}, \eta, \mu)$ and P^{\circledast} the value in the S -sorted mapping $P: Y \longrightarrow \mathbb{T}_{\Sigma}(X)$ of the functor \mathbf{T}_{Σ} ; and $Q^{\mathbf{A}} \circ P^{\mathbf{A}}$ the mapping from A_X to A_Z which sends an S -sorted mapping $u: X \longrightarrow A$ to the S -sorted mapping

$$(u^{\sharp} \circ P)^{\sharp} \circ Q: Z \longrightarrow A,$$

to show that $(Q \diamond P)^{\mathbf{A}} = Q^{\mathbf{A}} \circ P^{\mathbf{A}}$ it is enough to prove that the Σ -homomorphisms $u^{\sharp} \circ \mu_X \circ P^{\circledast}$ and $(u^{\sharp} \circ P)^{\sharp}$ from $\mathbf{T}_{\Sigma}(Y)$ to \mathbf{A} are identical. But this follows from the equation

$$u^{\sharp} \circ \mu_X \circ P^{\circledast} \circ \eta_Y = (u^{\sharp} \circ P)^{\sharp} \circ \eta_Y,$$

that, in its turn, is a consequence of the laws for the monad \mathbb{T}_{Σ} and of the equation

$$P^{\circledast} \circ \eta_Y = \eta_{\mathbb{T}_{\Sigma}(X)} \circ P,$$

where η_Y is the canonical embedding of Y into $\mathbb{T}_{\Sigma}(Y)$ and $\eta_{\mathbb{T}_{\Sigma}(X)}$ the canonical embedding of $\mathbb{T}_{\Sigma}(X)$ into $\mathbb{T}_{\Sigma}(\mathbb{T}_{\Sigma}(X))$. \square

This lemma has as an immediate consequence the following

Corollary 4. *Let Σ be a signature and \mathbf{A} a Σ -algebra. Then there exists a functor $\mathrm{Tr}^{\Sigma, \mathbf{A}}$ from $\mathbf{Ter}(\Sigma)$ to \mathbf{Set} which sends an S -sorted set X to the set $\mathrm{Tr}^{\Sigma, \mathbf{A}}(X) = A_X$ and a term $P: X \longrightarrow Y$ to $\mathrm{Tr}^{\Sigma, \mathbf{A}}(P) = P^{\mathbf{A}}: A_X \longrightarrow A_Y$, the term operation on \mathbf{A} determined by P .*

Therefore, from the definition of the object and morphism mappings of the functors of the type $\mathrm{Tr}^{\Sigma, \mathbf{A}}$, we see that they encapsulate the procedure of realization of terms. And, from the fact that they preserve identities and compositions in $\mathbf{Ter}(\Sigma)$, we conclude that they formally represent the two basic intuitions about the behaviour of the just named procedure, i.e., that the realization of an identity term is an identity term operation, and that the realization of a composite of two terms is the composite of their respective realizations (in the same order).

Remark. By identifying the Σ -algebras with the \mathbb{T}_{Σ} -algebras, the just stated corollary can be interpreted as meaning that every Σ -algebra is a functor from $\mathbf{Ter}(\Sigma) = \mathbf{Kl}(\mathbb{T}_{\Sigma})^{\mathrm{op}}$ to \mathbf{Set} .

Before stating the following lemma we recall that, for an S -sorted mapping f from an S -sorted set A into a like one B and an S -sorted set X , f_X is the mapping from A_X to B_X which assigns to an S -sorted mapping $u: X \longrightarrow A$ the S -sorted mapping $f \circ u$, i.e., f_X is the value at X of the natural transformation $\mathbb{H}(\cdot, f)$ from the contravariant functor $\mathbb{H}(\cdot, A)$ to the contravariant functor $\mathbb{H}(\cdot, B)$, both from $(\mathbf{Set}^S)^{\mathrm{op}}$ to \mathbf{Set} .

Lemma 2. *Let f be a Σ -homomorphism from \mathbf{A} to \mathbf{B} and P a term of type (X, Y) in $\mathbf{Ter}(\Sigma)$. Then the following diagram commutes*

$$\begin{array}{ccc}
 & A_X & \\
 f_X \swarrow & & \searrow P^{\mathbf{A}} \\
 B_X & & A_Y \\
 P^{\mathbf{B}} \searrow & & \swarrow f_Y \\
 & B_Y &
 \end{array}$$

We agree to denote by f_P the diagonal mapping from A_X to B_Y in the above commutative diagram.

Proof. Given an S -sorted mapping $u: X \longrightarrow A$, we have that $(f \circ u)^\# = f \circ u^\#$, by the universal property of the free Σ -algebra on X and taking into account that f is a Σ -homomorphism from \mathbf{A} to \mathbf{B} . Therefore, since $P^{\mathbf{B}} \circ f_X(u) = (f \circ u)^\# \circ P$, and $f_Y \circ P^{\mathbf{A}}(u) = f \circ (u^\# \circ P)$, we have that $P^{\mathbf{B}} \circ f_X(u) = f_Y \circ P^{\mathbf{A}}(u)$. Thus $P^{\mathbf{B}} \circ f_X = f_Y \circ P^{\mathbf{A}}$. \square

This lemma has as an immediate consequence the following

Corollary 5. *Let Σ be a signature and f a Σ -homomorphism from \mathbf{A} to \mathbf{B} . Then there exists a natural transformation $\mathrm{Tr}^{\Sigma, f}$ from the functor $\mathrm{Tr}^{\Sigma, \mathbf{A}}$ to the functor $\mathrm{Tr}^{\Sigma, \mathbf{B}}$, as reflected in the diagram*

$$\begin{array}{ccc}
 & \mathrm{Tr}^{\Sigma, \mathbf{A}} & \\
 \mathrm{Ter}(\Sigma) & \begin{array}{c} \curvearrowright \\ \Downarrow \mathrm{Tr}^{\Sigma, f} \\ \curvearrowleft \end{array} & \mathbf{Set} \\
 & \mathrm{Tr}^{\Sigma, \mathbf{B}} &
 \end{array}$$

which sends an S -sorted set X to the mapping $\mathrm{Tr}_X^{\Sigma, f} = f_X$ from A_X to B_X . Besides, we have that

- (1) For $\mathrm{id}_{\mathbf{A}}$, the identity Σ -homomorphism at \mathbf{A} , it is the case that

$$\mathrm{Tr}^{\Sigma, \mathrm{id}_{\mathbf{A}}} = \mathrm{id}_{\mathrm{Tr}^{\Sigma, \mathbf{A}}}.$$

- (2) If $g: \mathbf{B} \longrightarrow \mathbf{C}$ is another Σ -homomorphism, then

$$\mathrm{Tr}^{\Sigma, g \circ f} = \mathrm{Tr}^{\Sigma, g} \circ \mathrm{Tr}^{\Sigma, f}.$$

Therefore, the naturalness of the procedure of realization of terms as term operations on the different algebras is embodied in the natural transformations of the type $\mathrm{Tr}^{\Sigma, f}$.

Remark. By identifying the Σ -homomorphisms with the \mathbb{T}_{Σ} -homomorphisms, the just stated corollary can be interpreted as meaning that every Σ -homomorphism f from \mathbf{A} to \mathbf{B} is a natural transformation from the functor $\mathrm{Tr}^{\Sigma, \mathbf{A}}$ to the functor $\mathrm{Tr}^{\Sigma, \mathbf{B}}$, both from $\mathbf{Ter}(\Sigma) = \mathbf{Kl}(\mathbb{T}_{\Sigma})^{\mathrm{op}}$ to \mathbf{Set} . Actually, each homomorphism (\mathbf{d}, f) from an algebra (Σ, \mathbf{A}) into a like one (Λ, \mathbf{B}) is identifiable to a morphism (in the category $(\mathbf{Cat})_{//\mathbf{Set}}$, see [31], p. (sub) 186) from the object $(\mathbf{Ter}(\Sigma), \mathrm{Tr}^{\Sigma, \mathbf{A}})$ over \mathbf{Set} to the object $(\mathbf{Ter}(\Lambda), \mathrm{Tr}^{\Lambda, \mathbf{B}})$ over \mathbf{Set} , concretely, to the morphism given by

the pair $(\mathbf{d}_\circ, (\theta_{\cdot, B}^\varphi)^{-1} \circ \mathbf{H}(\cdot, f))$, and represented by the following diagram

$$\begin{array}{ccc}
 \mathbf{Ter}(\Sigma) & \xrightarrow{\mathbf{d}_\circ} & \mathbf{Ter}(\Lambda) \\
 \text{Tr}^{\Sigma, \mathbf{A}} \downarrow & \scriptstyle (\theta_{\cdot, B}^\varphi)^{-1} \circ \mathbf{H}(\cdot, f) \cong & \downarrow \text{Tr}^{\Lambda, \mathbf{B}} \\
 \mathbf{Set} & \xrightarrow{\text{Id}} & \mathbf{Set}
 \end{array}$$

where $\mathbf{H}(\cdot, f)$ is the natural transformation from the contravariant hom-functor $\mathbf{H}(\cdot, A)$ to the contravariant hom-functor $\mathbf{H}(\cdot, B_\varphi)$, and $(\theta_{\cdot, B}^\varphi)^{-1}$ the natural isomorphism from $\mathbf{H}(\cdot, B_\varphi)$ to $\mathbf{H}(\coprod_\varphi(\cdot), B)$. Observe that the naturalness of $(\theta_{\cdot, B}^\varphi)^{-1} \circ \mathbf{H}(\cdot, f)$ means that, for every term P for Σ of type (X, Y) , the following diagram commutes

$$\begin{array}{ccccc}
 A_X = \text{Hom}(X, A) & \xrightarrow{\mathbf{H}(X, f)} & \text{Hom}(X, B_\varphi) & \xrightarrow{(\theta_{X, B}^\varphi)^{-1}} & \text{Hom}(\coprod_\varphi X, B) = B_{\coprod_\varphi X} \\
 \downarrow P^{\mathbf{A}} & & & & \downarrow \mathbf{d}_\circ(P)^{\mathbf{B}} \\
 A_Y = \text{Hom}(Y, A) & \xrightarrow{\mathbf{H}(Y, f)} & \text{Hom}(Y, B_\varphi) & \xrightarrow{(\theta_{Y, B}^\varphi)^{-1}} & \text{Hom}(\coprod_\varphi Y, B) = B_{\coprod_\varphi Y}
 \end{array}$$

From the identification of the homomorphisms between algebras in the category \mathbf{Alg} to some convenient morphisms between the associated objects over \mathbf{Set} , we can conclude, e.g., that the concept of homomorphism as defined by Bénabou in [2] (that does not allow the variation of the signature and therefore it works between algebras of the same signature (see [2], p. (sub) 16, last paragraph)), corresponds itself, for a signature Σ and a Σ -homomorphism f from \mathbf{A} to \mathbf{B} , to the (very special) case in which $(\mathbf{d}_\circ, (\theta_{\cdot, B}^\varphi)^{-1} \circ \mathbf{H}(\cdot, f))$ is precisely

$$(\mathbf{d}_\circ, (\theta_{\cdot, B}^\varphi)^{-1} \circ \mathbf{H}(\cdot, f)) = (\text{Id}_{\mathbf{Ter}(\Sigma)}, \mathbf{H}(\cdot, f)),$$

that we represent by the following diagram

$$\begin{array}{ccc}
 \mathbf{Ter}(\Sigma) & \xrightarrow{\text{Id}} & \mathbf{Ter}(\Sigma) \\
 \text{Tr}^{\Sigma, \mathbf{A}} \downarrow & \scriptstyle \mathbf{H}(\cdot, f) \cong & \downarrow \text{Tr}^{\Sigma, \mathbf{B}} \\
 \mathbf{Set} & \xrightarrow{\text{Id}} & \mathbf{Set}
 \end{array}$$

i.e., definitely, it corresponds to the natural transformation $\text{Tr}^{\Sigma, f}$ from the functor $\text{Tr}^{\Sigma, \mathbf{A}}$ to the functor $\text{Tr}^{\Sigma, \mathbf{B}}$.

For an arbitrary, but fixed, signature Σ the family of functors $(\text{Tr}^{\Sigma, \mathbf{A}})_{\mathbf{A} \in \mathbf{Alg}(\Sigma)}$ together with the family of natural transformations $(\text{Tr}^{\Sigma, f})_{f \in \text{Mor}(\mathbf{Alg}(\Sigma))}$ actually constitute the object and morphism mappings, respectively, of a functor $\text{Tr}^{\Sigma, (\cdot)}$ from the category $\mathbf{Alg}(\Sigma)$ to the exponential category $\mathbf{Set}^{\mathbf{Ter}(\Sigma)}$. And it is precisely the functor $\text{Tr}^{\Sigma, (\cdot)}$ that will allow us to prove, in the following proposition, the existence of a functor Tr^Σ from $\mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Sigma)$ to \mathbf{Set} that formalizes the realization of terms as term operations on algebras, but taking into account the variation of the algebras through the homomorphisms between them.

Proposition 18. *Let Σ be a signature. Then there exists a functor Tr^Σ from $\mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Sigma)$ to \mathbf{Set} defined as follows*

- (1) Tr^Σ sends a pair (\mathbf{A}, X) , formed by a Σ -algebra \mathbf{A} and an S -sorted set X , to the set $\text{Tr}^\Sigma(\mathbf{A}, X) = \text{Tr}^{\Sigma, \mathbf{A}}(X) = A_X$ of the S -sorted mappings from X to the underlying S -sorted set A of \mathbf{A} .
- (2) Tr^Σ sends an arrow (f, P) from (\mathbf{A}, X) to (\mathbf{B}, Y) in $\mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Sigma)$, to the mapping $\text{Tr}^\Sigma(f, P) = f_P$ from A_X to B_Y , i.e., to $\text{Tr}^{\Sigma, \mathbf{B}}(P) \circ \text{Tr}_X^{\Sigma, f} = \text{Tr}_Y^{\Sigma, f} \circ \text{Tr}^{\Sigma, \mathbf{A}}(P)$.

Proof. We restrict ourselves to prove that Tr^Σ preserves compositions. Given two arrows (f, P) from (\mathbf{A}, X) to (\mathbf{B}, Y) and (g, Q) from (\mathbf{B}, Y) to (\mathbf{C}, Z) , we have that $(g \circ f)_{Q \circ P}$ is the diagonal mapping in the commutative diagram

$$\begin{array}{ccccc}
 & & A_X & & \\
 & (g \circ f)_X & \swarrow & & \searrow (Q \circ P)^{\mathbf{A}} \\
 & C_X & & (g \circ f)_{Q \circ P} & & A_Z \\
 & & \searrow (Q \circ P)^{\mathbf{C}} & & \swarrow (g \circ f)_Z \\
 & & & C_Z & &
 \end{array}$$

But it happens that the following diagram commutes

$$\begin{array}{ccccccc}
 & & & & A_X & & \\
 & & & & \swarrow f_X & & \searrow P^{\mathbf{A}} \\
 & & & & B_X & & A_Y \\
 & & & & \swarrow g_X & & \searrow Q^{\mathbf{A}} \\
 & & & & C_X & & A_Z \\
 & & & & \swarrow P^{\mathbf{C}} & & \searrow f_Z \\
 & & & & C_Y & & B_Z \\
 & & & & \swarrow Q^{\mathbf{C}} & & \searrow g_Z \\
 & & & & C_Z & &
 \end{array}$$

Therefore, by Lemma 1, we have that $g_Q \circ f_P = (g \circ f)_{Q \circ P}$. \square

Remark. The functor Tr^Σ can also be obtained from the functor $\text{Tr}^{\Sigma, (\cdot)}$ as the composite of the functors $\text{Tr}^{\Sigma, (\cdot)} \times \text{Id}_{\mathbf{Ter}(\Sigma)}$ and $\text{Ev}_{\mathbf{Ter}(\Sigma), \mathbf{Set}}$, as in the following diagram

$$\begin{array}{ccc}
 \mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Sigma) & & \\
 \downarrow \text{Tr}^{\Sigma, (\cdot)} \times \text{Id}_{\mathbf{Ter}(\Sigma)} & \searrow \text{Tr}^\Sigma & \\
 \mathbf{Set}^{\mathbf{Ter}(\Sigma)} \times \mathbf{Ter}(\Sigma) & \xrightarrow{\text{Ev}_{\mathbf{Ter}(\Sigma), \mathbf{Set}}} & \mathbf{Set}
 \end{array}$$

where $\text{Ev}_{\mathbf{Ter}(\Sigma), \mathbf{Set}}$ is the evaluation functor, because of the natural isomorphism

$$\text{Hom}(\mathbf{Alg}(\Sigma), \mathbf{Set}^{\mathbf{Ter}(\Sigma)}) \cong \text{Hom}(\mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Sigma), \mathbf{Set}).$$

To accomplish the earlier stated second goal, i.e., to show the invariant character of the procedure of realization of terms under signature change, we prove, for a morphism $\mathbf{d}: \Sigma \longrightarrow \Lambda$, the existence of a natural isomorphism between two functors from $\mathbf{Alg}(\Lambda) \times \mathbf{Ter}(\Sigma)$ to \mathbf{Set} , constructed from the functors Tr^Λ , Tr^Σ , \mathbf{d}_\diamond and \mathbf{d}^* .

- (1) The contravariant functor \mathbf{Alg} from \mathbf{Sig} to \mathbf{Cat} , giving the category of models of a given signature,
- (2) The pseudo-functor \mathbf{Ter} from \mathbf{Sig} to \mathbf{Cat} , giving the dual of the Kleisli category for the standard monad derived from the adjunction induced by a given signature,
- (3) The family of functors $\mathbf{Tr} = (\mathbf{Tr}^\Sigma)_{\Sigma \in \mathbf{Sig}}$, where, for every signature Σ , \mathbf{Tr}^Σ is a functor from $\mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Sigma)$ to \mathbf{Set} that formalize the realization of terms as term operations on algebras, and
- (4) The family of natural isomorphisms $\theta = (\theta^{\mathbf{d}})_{\mathbf{d} \in \mathbf{Mor}(\mathbf{Sig})}$, where, for every signature morphism \mathbf{d} , $\theta^{\mathbf{d}}$ is the natural isomorphism that explains the invariant character of the procedure of realization of terms under the variation of the signature.

Our next goal is to construct the many-sorted term institution by combining adequately the above components. To attain the just stated goal we need to recall beforehand some auxiliary category-theoretic concepts. In particular, we proceed to define next, among others, the concept of pseudo-extranatural transformation in 2-categories and for pseudo-functors. This generality is necessary because, later on (e.g., in the sixth section), after adding a certain type of 2-cells (the transformations of Fujiwara) to the category $\mathbf{Sig}_{\mathbf{pd}}$, of signatures and polyderivors (defined in the fifth section), that converts it into a 2-category, still denoted by the same symbol, we will get another 2-category, precisely $\mathbf{Sig}_{\mathbf{pd}}^{\text{op}} \times \mathbf{Sig}_{\mathbf{pd}}$, on which we will define a pseudo-functor related by a pseudo-extranatural transformation to a definite functor (also defined on the same derived 2-category), from which we will get the so-called term 2-institution of Fujiwara.

Definition 8. Let \mathbf{C} and \mathbf{D} be two 2-categories, $F, G: \mathbf{C}^{\text{op}} \times \mathbf{C} \longrightarrow \mathbf{D}$ two 2-functors, and (α, β) a pair such that

- (1) For every 0-cell c in \mathbf{C} , $\alpha_c: F(c, c) \longrightarrow G(c, c)$ is a 1-cell in \mathbf{D} .
- (2) For every 1-cell $f: c \longrightarrow c'$ in \mathbf{C} , β_f is a 2-cell in \mathbf{D} from $G(1, f) \circ \alpha_c \circ F(f, 1)$ to $G(f, 1) \circ \alpha_{c'} \circ F(1, f)$.

Then we say that (α, β) is a

- (1) *Lax-dinatural transformation* from F to G if, for every 2-cell $\xi: f \Rightarrow g$ in \mathbf{C} , we have that

$$\beta_g \circ (G(1, \xi) * \alpha_c * F(\xi, 1)) = (G(\xi, 1) * \alpha_{c'} * F(1, \xi)) \circ \beta_f.$$

- (2) *Pseudo-dinatural transformation* from F to G if it is a lax-dinatural transformation and, for every $f: c \longrightarrow c'$ in \mathbf{C} , β_f is an isomorphism.
- (3) *2-dinatural transformation* from F to G if it is a lax-dinatural transformation and, for every $f: c \longrightarrow c'$ in \mathbf{C} , β_f is an identity.

The dinatural transformations when F and G are pseudo-functors will also be relevant for us. In this case it is necessary to impose additional conditions of compatibility with the natural isomorphisms of the pseudo-functors. The definition is as follows.

Definition 9. Let \mathbf{C} and \mathbf{D} be two 2-categories, (F, γ^F, ν^F) and (G, γ^G, ν^G) two pseudo-functors from $\mathbf{C}^{\text{op}} \times \mathbf{C}$ to \mathbf{D} , and (α, β) a pair such that

- (1) For every 0-cell c in \mathbf{C} , $\alpha_c: F(c, c) \longrightarrow G(c, c)$ is a 1-cell in \mathbf{D} .
- (2) For every 1-cell $f: c \longrightarrow c'$ in \mathbf{C} , β_f is a 2-cell in \mathbf{D} from $G(1, f) \circ \alpha_c \circ F(f, 1)$ to $G(f, 1) \circ \alpha_{c'} \circ F(1, f)$.

Then we say that (α, β) is a *lax-dinatural transformation* from (F, γ^F, ν^F) to (G, γ^G, ν^G) if it satisfies the following compatibility conditions:

(1) For every 2-cell $\xi: f \Rightarrow g$ in \mathbf{C} , we have that

$$\beta_g \circ (G(1, \xi) * \alpha_c * F(\xi, 1)) = (G(\xi, 1) * \alpha_{c'} * F(1, \xi)) \circ \beta_f.$$

(2) For every pair of 1-cells $f: c \longrightarrow c'$, $g: c' \longrightarrow c''$ in \mathbf{C} , we have that

$$\begin{aligned} \gamma_{(1,f),(1,g)}^F \circ (G(f, 1) * \beta_g * F(1, f)) \circ (G(1, g) * \beta_f * F(g, 1)) \\ = \beta_{g \circ f} \circ (\gamma_{(1,f),(1,g)}^G * \alpha_c * \gamma_{(g,1),(f,1)}^F). \end{aligned}$$

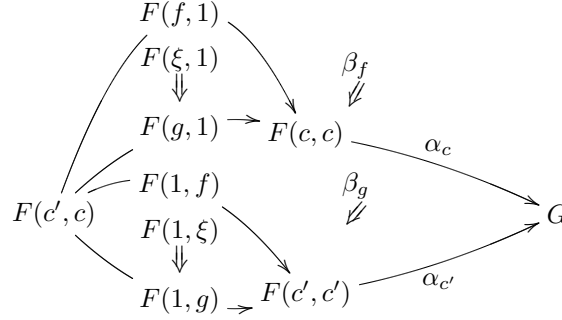
(3) For every object c in \mathbf{C} , we have that

$$\alpha_c * \nu_{(c,c)}^F = \nu_{(c,c)}^G * \alpha_c.$$

If the pseudo-functor G is independent of both variables, then we say that the above transformations are *lax-extranatural*, *pseudo-extranatural* or *extranatural*, respectively. Then the compatibility with the 2-cells of \mathbf{C} is equivalent to

$$\beta_g \circ (\alpha_c * F(\xi, 1)) = (\alpha_{c'} * F(1, \xi)) \circ \beta_f,$$

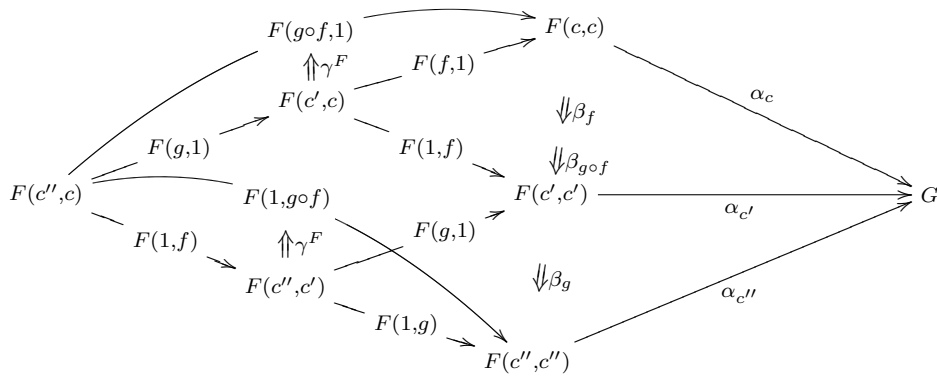
or geometrically:



and the compatibility of the composition of 1-cells in \mathbf{C} with the natural isomorphisms of F is equivalent to

$$\gamma_{(1,f),(1,g)}^F \circ (\beta_g * F(1, f)) \circ (\beta_f * F(g, 1)) = \beta_{g \circ f} \circ (\alpha_c * \gamma_{(g,1),(f,1)}^F),$$

or geometrically:



In the next proposition we construct a pseudo-functor $\text{Alg}(\cdot) \times \text{Ter}(\cdot)$ from the product category $\mathbf{Sig}^{\text{op}} \times \mathbf{Sig}$ to \mathbf{Cat} (obtained from the contravariant functor Alg and the pseudo-functor Ter), and prove that the family $\text{Tr} = (\text{Tr}^\Sigma)_{\Sigma \in \mathbf{Sig}}$, together with the family $\theta = (\theta^{\mathbf{d}})_{\mathbf{d} \in \text{Mor}(\mathbf{Sig})}$ is a pseudo-extranatural transformation from the pseudo-functor $\text{Alg}(\cdot) \times \text{Ter}(\cdot)$ to the functor $\mathbf{K}_{\mathbf{Set}}$ (from $\mathbf{Sig}^{\text{op}} \times \mathbf{Sig}$ to \mathbf{Cat}) that is constantly \mathbf{Set} .

defined by Feferman in [20], pp. 155–156, as a system

$$L = (\text{Typ}_L, \text{Str}_L, \text{Stc}_L, \models_L)$$

where Typ_L is a non-empty set of similarity types, called the *admitted types* of L , and $\text{Str}_L, \text{Stc}_L, \models_L$ are functions with domain Typ_L such that for each admitted types τ, τ' :

- (i) $\text{Str}_L(\tau)$ is a sub-collection of $\text{Str}(\tau)$, called the *admitted structures for $L(\tau)$* ,
- (ii) $\text{Stc}_L(\tau)$ is a collection called the *sentences of $L(\tau)$* ,
- (iii) $\models_{L,\tau}$ is a sub-relation of $\text{Str}_L(\tau) \times \text{Stc}_L(\tau)$, called the *satisfaction (or truth) relation of $L(\tau)$* ,
- (iv) **Expansion.** $\tau \subseteq \tau' \Rightarrow \text{Stc}_L(\tau) \subseteq \text{Stc}_L(\tau')$; $\mathfrak{M}' \in \text{Str}_L(\tau') \Rightarrow \mathfrak{M}' \upharpoonright \tau \in \text{Str}_L(\tau)$ and $\varphi \in \text{Stc}_L(\tau) \Rightarrow [\mathfrak{M}' \upharpoonright \tau \models \varphi \Leftrightarrow \mathfrak{M}' \models \varphi]$,
- (v) **Renaming.** Each $\tau \equiv_\gamma \tau'$ induces a 1 – 1 correspondence

$$\bar{\gamma}: \text{Stc}(\tau) \longrightarrow \text{Stc}(\tau')$$

such that if $\mathfrak{M} \in \text{Str}_L(\tau)$ and $\mathfrak{M}' \in \text{Str}(\tau')$ and $\mathfrak{M} \equiv_\gamma \mathfrak{M}'$, then $\mathfrak{M}' \in \text{Str}_L(\tau')$ and $\mathfrak{M} \models \varphi \Leftrightarrow \mathfrak{M}' \models \bar{\gamma}(\varphi)$, and

- (vi) **Isomorphism.** If $\mathfrak{M} \in \text{Str}_L(\tau)$ and $\mathfrak{M}' \in \text{Str}(\tau)$ and $\mathfrak{M} \cong \mathfrak{M}'$, then $\mathfrak{M}' \in \text{Str}_L(\tau)$ and $\mathfrak{M} \models \varphi \Leftrightarrow \mathfrak{M}' \models \varphi$;

and that of a *logic* \mathfrak{L}^* defined by Barwise in [1], pp. 234–235, where he says that it consists of a syntax and a semantics which fit together nicely. The *syntax of \mathfrak{L}^** is a functor $*$ on some category \mathbf{C} of languages to the category of classes. The functor $*$ satisfies the following axiom:

Occurrence Axiom. For every \mathfrak{L}^* -sentence φ there is a smallest (under \subseteq) language L_φ in \mathbf{C} such that $\varphi \in L_\varphi^*$. If $i: L_\varphi \subseteq K$ is an inclusion morphism, so is $i^*: L_\varphi^* \subseteq K^*$.

The *semantics of \mathfrak{L}^** is a relation \models such that if $\mathfrak{M} \models \varphi$, then \mathfrak{M} is an L -structure for some L in \mathbf{C} and $\varphi \in L^*$. It satisfies the following axiom:

Isomorphism Axiom. If $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \cong \mathfrak{N}$, then $\mathfrak{N} \models \varphi$.

The syntax and semantics of \mathfrak{L}^* fit together according to the final axiom.

Translation Axiom. For every \mathfrak{L}^* sentence φ , every K -structure \mathfrak{M} and every morphism $\alpha: L_\varphi \longrightarrow K$

$$\mathfrak{M} \models \alpha^*(\varphi) \text{ iff } \mathfrak{M}^{-\alpha} \text{ is an } L_\varphi\text{-structure and } \mathfrak{M}^{-\alpha} \models \varphi.$$

We recall that Goguen and Burstall in [24], p. 229, define an *institution* as a category **Sign**, of signatures, a functor Sen from **Sign** to **Set**, giving the set of *sentences* over a given signature, a functor Mod from **Sign** to \mathbf{Cat}^{op} , giving the category of *models* of a given signature, and, for each $\Sigma \in \mathbf{Sign}$, a satisfaction relation $\models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times \text{Sen}(\Sigma)$, where $|\cdot|$ is the endofunctor of **Cat** which sends a category to the discrete category on its set of objects, such that, for each morphism $\varphi: \Sigma \longrightarrow \Sigma'$, the

Satisfaction Condition. $\mathbf{M}' \models_{\Sigma'} \varphi(e)$ iff $\varphi(\mathbf{M}') \models_\Sigma e$,

holds for each $\mathbf{M}' \in |\mathbf{Mod}(\Sigma')|$ and each $e \in \text{Sen}(\Sigma)$. Later on, in [25], p. 316, they define an *institution* as a category **Sign**, of signatures, a functor Sen from **Sign** to **Cat** (observe the large-scale change from **Set** to **Cat**), giving *sentences* and *proofs* over a given signature, a functor Mod from **Sign** to \mathbf{Cat}^{op} , giving the category of *models* of a given signature, and a satisfaction relation $\models_\Sigma \subseteq |\mathbf{Mod}(\Sigma)| \times |\mathbf{Sen}(\Sigma)|$, for each $\Sigma \in |\mathbf{Sign}|$, such that

Satisfaction Condition: $\mathbf{M}' \models_{\Sigma'} \text{Sen}(\varphi)s$ iff $\text{Mod}(\varphi)\mathbf{M}' \models_\Sigma s$, for each $\varphi: \Sigma \longrightarrow \Sigma'$ in **Sign**, $\mathbf{M}' \in |\mathbf{Mod}(\Sigma')|$ and $s \in |\mathbf{Sen}(\Sigma)|$, and

Soundness Condition: if $\mathbf{M} \models_{\Sigma} s$, then $\mathbf{M} \models_{\Sigma} s'$, for each $\mathbf{M} \in |\mathbf{Mod}(\Sigma)|$ and $s \longrightarrow s' \in \mathbf{Sen}(\Sigma)$.

Besides, the same authors, in [25], p. 327, define, for a category \mathbf{V} , a *generalized V-institution* as a pair of functors \mathbf{Mod} , from $\mathbf{Sign}^{\text{op}}$ to \mathbf{Cat} , and \mathbf{Sen} , from \mathbf{Sign} to \mathbf{Cat} , with an extranatural transformation \models from $|\mathbf{Mod}(\cdot)| \times \mathbf{Sen}(\cdot)$ to \mathbf{V} . Observe that the second concept of institution falls under this last one because, taking as \mathbf{V} the category $\mathbf{2}$, with two objects and just one morphism not the identity, the existence of an extranatural transformation from $|\mathbf{Mod}(\cdot)| \times \mathbf{Sen}(\cdot)$ to $\mathbf{2}$ is equivalent to the above satisfaction and soundness conditions.

But it happens that terms and algebras are not only compatible with signature changes, but also with the category structure on the terms and the algebras. From this it follows that the restriction imposed by Goguen and Burstall in [25], p. 327, to the concept of institution, concretely, that the domain of the extranatural transformation is $|\mathbf{Mod}(\cdot)| \times \mathbf{Sen}(\cdot)$, is a real loss of generality, that prevents to reflect faithfully the involved complexity. Therefore such a restriction, at least in this case, is unsound and should be left out. Thus, under these circumstances, we propose the following definitions of 2-institution and institution, both relative to a given category.

Definition 10. Let \mathbf{C} be a category. Then a *2-institution on C* is a quadruple $(\mathbf{Sig}, \mathbf{Mod}, \mathbf{Sen}, (\alpha, \beta))$, where

- (1) \mathbf{Sig} is a 2-category.
- (2) $\mathbf{Mod}: \mathbf{Sig}^{\text{op}} \longrightarrow \mathbf{Cat}$ a pseudo-functor.
- (3) $\mathbf{Sen}: \mathbf{Sig} \longrightarrow \mathbf{Cat}$ a pseudo-functor.
- (4) $(\alpha, \beta): \mathbf{Mod}(\cdot) \times \mathbf{Sen}(\cdot) \longrightarrow \mathbf{K}_{\mathbf{C}}$ a pseudo-extranatural transformation.

If \mathbf{Sig} is an ordinary category, instead of a 2-category, then we will speak of an *institution on C*.

Remark. The concept of 2-institution is defined relative to a category, i.e., it has meaning for a 0-cell \mathbf{C} of the 2-category $\mathbf{Cat} = 1\text{-Cat}$ of categories, functors, and natural transformations between functors. Therefore, if it were necessary for some application, the concept of 3-institution ought to be defined relative to a 0-cell \mathbf{C} of the 3-category 2-Cat of 2-categories, 2-functors, 2-natural transformations and modifications between transformations, and so forth.

Actually, 2-institutions and institutions, if they are understood as pseudo-extranatural transformations, go beyond both the classical conception of semantical truth defined (mathematically for the first time, through a recursive definition of satisfaction of a formula in an arbitrary relational system by a valuation of the variables in the system) in Tarski and Vaught [59], p. 85, and the latest conception of institution in Goguen and Burstall [25], p. 327.

From the above it follows, immediately, the following

Corollary 6. *The quadruple $\mathfrak{Tm} = (\mathbf{Sig}, \mathbf{Alg}, \mathbf{Ter}, (\text{Tr}, \theta))$ is an institution on the category \mathbf{Set} , the so-called many-sorted term institution, or, to abbreviate, the term institution.*

3. MANY-SORTED SPECIFICATIONS AND MORPHISMS.

In this section we begin by defining, for a signature Σ , the concept of Σ -equation, but for the generalized terms defined in the preceding section, the binary relation of satisfaction between Σ -algebras and Σ -equations, and the semantical consequence operators Cn_{Σ} . Then, after extending the translation of generalized terms up to generalized equations, we prove the corresponding satisfaction condition, and define a pseudo-functor LEq which assigns (among others) to a signature Σ , the discrete

category associated to the set of all labelled Σ -equations, that enables us to get the many-sorted equational institution \mathcal{LEq} .

Following this, in order to show that the semantical consequence operators Cn_Σ , associated to the different signatures Σ , are the components of a pseudo-functor, Cn , from the category of signatures to a convenient 2-category of monads, we define, by means of the construction of Ehresmann-Grothendieck, the category of many-sorted closure spaces. Then we prove that the pseudo-functor Cn , in its turn, is part of an entailment system, but for a concept of entailment system that generalizes that defined by Meseguer in [47], pp. 282–283.

After this we define, for the generalized terms, the concepts of many-sorted specification and of many-sorted specification morphism, from which we get the corresponding category, denoted by \mathbf{Spf} . Then by extending some of the notions and constructions previously developed for the category \mathbf{Sig} up to the category \mathbf{Spf} , we get \mathcal{Spf} , the many-sorted specification institution on \mathbf{Set} . Besides, we prove that there exists a morphism from \mathcal{Spf} to \mathcal{Tm} , the many-sorted term institution on \mathbf{Set} , which, together with the canonical embedding of \mathcal{Tm} into \mathcal{Spf} , makes of \mathcal{Tm} a retract of \mathcal{Spf} . We point out that, conveniently generalized, the many-sorted specification morphisms will be used, together with some other concepts, in the last section, to prove the equivalence between the many-sorted specifications of Hall and Bénabou.

We now define the equations over a given signature through the morphisms of the category of terms for the signature, what it means for an equation to be valid in an algebra, and the consequence operator on the ms-set of the equations.

Definition 11. Let Σ be a signature, X, Y two S -sorted sets and \mathbf{A} a Σ -algebra.

- (1) A Σ -equation of type (X, Y) is a pair $(P, Q): X \longrightarrow Y$ of parallel morphisms in $\mathbf{Ter}(\Sigma) = \mathbf{Kl}(\mathcal{T}_\Sigma)^{\text{op}}$, hence $(P, Q) \in \text{Hom}(Y, \mathcal{T}_\Sigma(X))^2$, and a Σ -equation is a Σ -equation of type (X, Y) for some S -sorted sets X, Y . We denote by $\text{Eq}(\Sigma)$ the $(\mathcal{U}^S)^2$ -sorted set of the Σ -equations, i.e., the family $(\text{Hom}(Y, \mathcal{T}_\Sigma(X))^2)_{X, Y \in \mathcal{U}^S}$.
- (2) We say that a Σ -equation $(P, Q): X \longrightarrow Y$ is *valid* in \mathbf{A} (or that \mathbf{A} is a *model* of (P, Q) , or also that \mathbf{A} *satisfies* (P, Q)), $\mathbf{A} \models_{X, Y}^\Sigma (P, Q)$, iff, for every $s \in S$ and $y \in Y_s$, $\mathbf{A} \models_{X, s}^\Sigma (P_s(y), Q_s(y))$, i.e., $(P_s(y))^\mathbf{A} = (Q_s(y))^\mathbf{A}$. We extend this satisfaction relation between Σ -algebras and Σ -equations to Σ -algebras \mathbf{A} and families $\mathcal{E} \subseteq \text{Eq}(\Sigma)$ by agreeing that $\mathbf{A} \models^\Sigma \mathcal{E}$ iff, for every $X, Y \in \mathcal{U}^S$ and $(P, Q) \in \mathcal{E}_{X, Y}$, we have that $\mathbf{A} \models_{X, Y}^\Sigma (P, Q)$.
- (3) We denote by Cn_Σ the operator on the $(\mathcal{U}^S)^2$ -sorted set $\text{Eq}(\Sigma)$ which assigns to $\mathcal{E} \subseteq \text{Eq}(\Sigma)$ the $(\mathcal{U}^S)^2$ -sorted set $\text{Cn}_\Sigma(\mathcal{E})$, where, for every $X, Y \in \mathcal{U}^S$ and $(P, Q) \in \text{Eq}(\Sigma)_{X, Y}$, $(P, Q) \in \text{Cn}_\Sigma(\mathcal{E})_{X, Y}$ iff, for every Σ -algebra \mathbf{A} , if $\mathbf{A} \models^\Sigma \mathcal{E}$, then $\mathbf{A} \models_{X, Y}^\Sigma (P, Q)$. We call $\text{Cn}_\Sigma(\mathcal{E})$ the $(\mathcal{U}^S)^2$ -sorted set of the *semantical consequences* of \mathcal{E} .

If we keep in mind that for a term $P: X \longrightarrow Y$ for Σ of type (X, Y) , $P^\mathbf{A}$, the term operation on \mathbf{A} determined by P , is the mapping from A_X to A_Y which assigns to an S -sorted mapping $f: X \longrightarrow A$ precisely $f^\sharp \circ P: Y \longrightarrow A$, then we get the following convenient characterization of the relation $\mathbf{A} \models_{X, Y}^\Sigma (P, Q)$:

$$\mathbf{A} \models_{X, Y}^\Sigma (P, Q) \quad \text{iff} \quad P^\mathbf{A} = Q^\mathbf{A}.$$

Besides, by the Completeness Theorem in [13], for $\mathcal{E} \subseteq \text{Eq}(\Sigma)$, we have that $\text{Cn}_\Sigma(\mathcal{E})$ is precisely $\text{Cg}_{\mathbf{Ter}(\Sigma)}^\Pi(\mathcal{E})$, i.e., the smallest Π -compatible congruence on $\mathbf{Ter}(\Sigma)$ that contains \mathcal{E} , where the superscript Π in the operator $\text{Cg}_{\mathbf{Ter}(\Sigma)}^\Pi$ abbreviates “product”. Therefore the operator Cn_Σ on the $(\mathcal{U}^S)^2$ -sorted set $\text{Eq}(\Sigma)$ is a closure operator.

Remark. It is true that, for a signature Σ , in order to equationally characterize the varieties (resp., the finitary varieties) of Σ -algebras it is enough to consider the S -finite (resp., the finite) subsets of an arbitrary, but fixed, S -sorted set V^S with a countable infinity of variables in each coordinate. However, the generalized terms and equations proposed in this paper, besides containing as particular cases the ordinary terms and equations, respectively, have proved their worth, e.g., in the proof of the Completeness Theorem for monads in categories of sorted sets in [13], and also to attain a truly category-theoretical understanding of the subject matter (through the theory of monads as sketched at the end of the sixth section). Moreover, the generalized terms and equations have the advantage over the ordinary terms and equations of being automatically dualizable, thus allowing the definition of the generalized coterms and coequations, from which it is easily obtainable, e.g., a Completeness Theorem for comonads in categories of sorted sets.

By recalling that every signature morphism \mathbf{d} from Σ to Λ determines a functor \mathbf{d}_\diamond from $\mathbf{Ter}(\Sigma)$ to $\mathbf{Ter}(\Lambda)$, and taking into account the above definition of the equations for a signature, we next formalize the procedure of translation, by means of a signature morphism, of equations for a signature into equations for another signature in the following

Definition 12. Let $\mathbf{d}: \Sigma \longrightarrow \Lambda$ be a signature morphism. Then \mathbf{d} induces an ms-mapping

$$((\coprod_\varphi)^2, \mathbf{d}_\diamond^2): ((\mathbf{U}^S)^2, \text{Eq}(\Sigma)) \longrightarrow ((\mathbf{U}^T)^2, \text{Eq}(\Lambda)),$$

the so called *translation of equations for Σ into equations for Λ relative to \mathbf{d}* , where

- (1) $(\coprod_\varphi)^2$ is the mapping from $(\mathbf{U}^S)^2$ to $(\mathbf{U}^T)^2$ which sends a pair of S -sorted sets (X, Y) to the pair $(\coprod_\varphi X, \coprod_\varphi Y)$ of T -sorted sets, and
- (2) \mathbf{d}_\diamond^2 the $(\mathbf{U}^S)^2$ -sorted mapping which to a Σ -equation (P, Q) of type (X, Y) assigns the Λ -equation $(\mathbf{d}_\diamond(P), \mathbf{d}_\diamond(Q))$ of type $(\coprod_\varphi X, \coprod_\varphi Y)$.

Once defined the translation of equations, we prove in the following lemma the invariance of the relation of satisfaction under signature change, also known, for those following the terminology coined by Goguen and Burstall in [24], p. 229, as the *satisfaction condition*.

Lemma 3. Let $\mathbf{d}: \Sigma \longrightarrow \Lambda$ be a signature morphism, (P, Q) a Σ -equation of type (X, Y) and \mathbf{A} a Λ -algebra. Then we have that

$$\mathbf{d}^*(\mathbf{A}) \models_{X, Y}^\Sigma (P, Q) \text{ iff } \mathbf{A} \models_{\coprod_\varphi X, \coprod_\varphi Y}^\Lambda (\mathbf{d}_\diamond(P), \mathbf{d}_\diamond(Q)).$$

Proof. The condition $\mathbf{d}^*(\mathbf{A}) \models_{X, Y}^\Sigma (P, Q)$ is equivalent to $P^{\mathbf{d}^*(\mathbf{A})} = Q^{\mathbf{d}^*(\mathbf{A})}$ but this condition, by Proposition 16, is equivalent to $\mathbf{d}_\diamond(P)^{\mathbf{A}} = \mathbf{d}_\diamond(Q)^{\mathbf{A}}$, therefore it is also equivalent to the condition $\mathbf{A} \models_{\coprod_\varphi X, \coprod_\varphi Y}^\Lambda (\mathbf{d}_\diamond(P), \mathbf{d}_\diamond(Q))$. \square

Related to the quasi-triviality of the (short and conceptual) proof of Lemma 3 (as a consequence, essentially, of the fact that it is ultimately rooted in Proposition 16), perhaps, it would be convenient to recall that Goguen and Burstall, in [24], p. 228, have omitted the corresponding proof because they qualify it as being not entirely trivial.

Everything we have made up to this point suggest that the many-sorted term institution introduced in the preceding section, if an institution is understood as meaning a pseudo-extranatural transformation, neither is a useless institution nor a trivial step in a natural process of evolution of the concept of institution, by adaptation to situations not noticed explicitly until now. And this can indicate that it is not unreasonable to give support to the view that the many-sorted term

institution \mathfrak{Im} is, because of its primitivity and elementariness, in fact, more basic, or fundamental, than the many-sorted equational institution, defined immediately below.

To construct the many-sorted equational institution we now define a pseudo-functor \mathbf{LEq} on the category of signatures. In order to do so (and also to define, later on, another pseudo-functor that will contribute to the construction of a certain entailment system founded on the family of consequences), we need to assume, besides the Grothendieck universe \mathcal{U} , the existence of another one \mathcal{V} such that $\mathcal{U} \in \mathcal{V}$. The new Grothendieck universe \mathcal{V} will be used to construct the appropriate 2-categories where the just named pseudo-functors will take its values. Therefore, to exclude any misunderstanding, we agree to denote those categories \mathbf{C} properly depending on \mathcal{V} by $\mathbf{C}_{\mathcal{V}}$. However, since the additional assumption of a universe \mathcal{V} such that $\mathcal{U} \in \mathcal{V}$, will be used, almost, exclusively in this section, we do not label those categories depending on \mathcal{U} with the subscript \mathcal{U} , such as has been done until now.

Definition 13. We denote by \mathbf{LEq} the pseudo-functor from \mathbf{Sig} to $\mathbf{Cat}_{\mathcal{V}}$ given by the following data

- (1) The object mapping of \mathbf{LEq} is that which sends a signature Σ to the discrete category $\mathbf{LEq}(\Sigma)$ canonically associated to the set

$$\bigcup_{X, Y \in \mathcal{U}} (\text{Hom}(Y, T_{\Sigma}(X)))^2 \times \{(X, Y)\}$$

of *labelled Σ -equations*, i.e., the set of all pairs $((P, Q), (X, Y))$ with (P, Q) a Σ -equation of type (X, Y) , for some $X, Y \in \mathcal{U}$.

- (2) The morphism mapping of \mathbf{LEq} is that which sends a signature morphism \mathbf{d} from Σ to Λ to the functor $\mathbf{LEq}(\mathbf{d})$ from $\mathbf{LEq}(\Sigma)$ to $\mathbf{LEq}(\Lambda)$ which assigns to the labelled equation $((P, Q), (X, Y))$ in $\mathbf{LEq}(\Sigma)$ the labelled equation

$$\mathbf{LEq}(\mathbf{d})((P, Q), (X, Y)) = ((\mathbf{d}_{\circ}(P), \mathbf{d}_{\circ}(Q)), (\coprod_{\varphi} X, \coprod_{\varphi} Y))$$

in $\mathbf{LEq}(\Lambda)$.

Corollary 7. *The quadruple $\mathfrak{LEq} = (\mathbf{Sig}, \text{Alg}, \mathbf{LEq}, (\models, \theta))$ is an institution on $\mathbf{2}$, the so-called many-sorted equational institution, or, to abbreviate, the equational institution.*

Before examining the many-sorted specifications and the many-sorted specification morphisms, it seems worthwhile to give a category-theoretic look at the concept of equational consequence. Concretely, what we want to establish now is the following:

- (1) That the closure operators Cn_{Σ} are, essentially, the components of a suitable pseudo-functor Cn from the category \mathbf{Sig} , of signatures, to a convenient 2-category $\mathbf{Mnd}_{\mathcal{V}, \text{alg}}$, of monads, for a Grothendieck universe \mathcal{V} such that $\mathcal{U} \in \mathcal{V}$, obtained by properly choosing the 2-cells in the 2-category $\mathbf{Mnd}_{\mathcal{V}}$, of monads for \mathcal{V} , and
- (2) That the pseudo-functor Cn is in fact part of an entailment system (understood in a more general sense than that defined by Meseguer in [47], pp. 282–283).

However, in order to do what has been just enumerated, we should begin by a thorough investigation of the building blocks that are involved and constitute the basis for the understanding of the equational consequence from the category-theoretical standpoint. Explicitly, this means that we should carry through the following:

- (1) To assign to each set of sorts S the corresponding category $\mathbf{ClSp}(S)$, of S -sorted closure spaces, and to associate to an arbitrary mapping $\varphi: S \longrightarrow T$,

from a set of sorts S into a like one T , the corresponding functor $\Delta_\varphi^{\text{cl}}$ from $\mathbf{ClSp}(T)$ to $\mathbf{ClSp}(S)$, and all in such a way that both procedures give rise to a contravariant functor Δ^{cl} from \mathbf{Sig} to \mathbf{Cat} , and

- (2) To get the category \mathbf{MClSp} , of many-sorted closure spaces and morphism between them, by applying the construction of Ehresmann-Grothendieck to the contravariant functor Δ^{cl} .

After having completed the above, the pseudo-functor \mathbf{Cn} will be obtained by using the fact that the category \mathbf{MClSp} can be identified to a subcategory of the 2-category $\mathbf{Mnd}_{\mathbf{V}, \text{alg}}$.

Definition 14. Let A be an S -sorted set.

- (1) An S -closure system on A is a subset \mathcal{C} of $\text{Sub}(A)$, the set of all S -sorted sets X such that, for every $s \in S$, $X_s \subseteq A_s$, abbreviated to $X \subseteq A$, that satisfies the following conditions
 - (a) $A \in \mathcal{C}$.
 - (b) For every $\mathcal{D} \subseteq \mathcal{C}$, if $\mathcal{D} \neq \emptyset$, then $\bigcap \mathcal{D} = (\bigcap_{D \in \mathcal{D}} D_s)_{s \in S} \in \mathcal{C}$.
 We denote by $\text{ClSy}(A)$ the set of the S -closure systems on A and by $\mathbf{ClSy}(A)$ the same set but ordered by inclusion. We call the pairs of the form (A, \mathcal{C}) , with $\mathcal{C} \in \text{ClSy}(A)$, S -closure system spaces.
- (2) An S -closure operator on A is an operator J on $\text{Sub}(A)$, i.e., a mapping from $\text{Sub}(A)$ to $\text{Sub}(A)$, such that, for every $X, Y \subseteq A$, satisfies the following conditions
 - (a) $X \subseteq J(X)$, i.e., J is extensive.
 - (b) If $X \subseteq Y$, then $J(X) \subseteq J(Y)$, i.e., J is isotone.
 - (c) $J(J(X)) = J(X)$, i.e., J is idempotent.

We denote by $\text{ClOp}(A)$ the set of the S -closure operators on A and by $\mathbf{ClOp}(A)$ the same set but ordered by the relation \leq , where, for J and K in $\text{ClOp}(A)$, we have that $J \leq K$ if, for every $X \subseteq A$, $J(X) \subseteq K(X)$. We call the pairs of the form (A, J) , with $J \in \text{ClOp}(A)$, S -closure operator spaces.

Example. For a Σ -algebra \mathbf{A} , the set $\text{Sub}(\mathbf{A})$, of subalgebras of \mathbf{A} , is an (algebraic) S -closure system on the S -sorted set A , and the operator $\text{Sg}_{\mathbf{A}}$, of generated subalgebra for \mathbf{A} , is an (algebraic) S -closure operator on A .

Example. For a Σ -algebra \mathbf{A} , the set $\text{Cgr}(\mathbf{A})$, of congruences on \mathbf{A} , is an (algebraic) S -closure system on the S -sorted set $A \times A = (A_s \times A_s)_{s \in S}$, and the operator $\text{Cg}_{\mathbf{A}}$, of generated congruence for \mathbf{A} , is an (algebraic) S -closure operator on $A \times A$.

Example. For a signature Σ , the operator Cn_{Σ} , of semantical consequence for Σ , is a $(\mathcal{U}^S)^2$ -closure operator on $\text{Eq}(\Sigma)$.

As in the single-sorted case, also in the many-sorted case, for a set of sorts S , every S -closure system \mathcal{C} on an S -sorted set A , when ordered by inclusion, induces a complete lattice $\mathcal{C} = (\mathcal{C}, \subseteq)$. Moreover, the ordered sets $\mathbf{ClOp}(A)$, of S -closure operators on A , and $\mathbf{ClSy}(A)$, of S -closure systems on A , are complete lattices and antiisomorphic through the mapping Fix from $\mathbf{ClOp}(A)$ to $\mathbf{ClSy}(A)$ which sends an S -closure operator J on A to the S -closure system $\text{Fix}(J) = \{X \subseteq A \mid J(X) = X\}$ on A , of the *fixed points of J* .

After having defined, for a set of sorts S , the objects of interest, i.e., in this case the S -closure system spaces and the S -closure operator spaces, we proceed to define the morphisms that are suitable for these mathematical constructs.

Definition 15. Let S be a set of sorts, A, B two S -sorted sets, \mathcal{C} and \mathcal{D} closure systems on A and B , respectively, and J and K closure operators on A and B , respectively.

- (1) An S -continuous mapping from the S -closure system space (A, \mathcal{C}) to the S -closure system space (B, \mathcal{D}) is a triple $((A, \mathcal{C}), f, (B, \mathcal{D}))$, denoted by $f: (A, \mathcal{C}) \longrightarrow (B, \mathcal{D})$, where f is an S -mapping from A to B such that, for every $D \in \mathcal{D}$, $f^{-1}[D] \in \mathcal{C}$.
- (2) An S -continuous mapping from the S -closure operator space (A, J) to the S -closure operator space (B, K) is a triple $((A, J), f, (B, K))$, denoted by $f: (A, J) \longrightarrow (B, K)$, where f is an S -mapping from A to B such that, for every $X \subseteq A$, $f[J(X)] \subseteq K[f[X]]$.

Example. For a Σ -homomorphism f from \mathbf{A} to \mathbf{B} and an S -sorted subset X of A , we have that $f[\text{Sg}_{\mathbf{A}}(X)] = \text{Sg}_{\mathbf{B}}(f[X])$. Therefore the Σ -homomorphism f induces an S -continuous (and closed) mapping from $(A, \text{Sg}_{\mathbf{A}})$ to $(B, \text{Sg}_{\mathbf{B}})$.

Example. For a Σ -homomorphism f from \mathbf{A} to \mathbf{B} and a congruence Ψ on \mathbf{B} , we have that $(f \times f)^{-1}[\Psi] \in \text{Cgr}(\mathbf{A})$. Therefore from the Σ -homomorphism f we get the S -continuous mapping $f \times f$ from $(A \times A, \text{Cgr}(\mathbf{A}))$ to $(B \times B, \text{Cgr}(\mathbf{B}))$.

Example. Later on, after having defined the category $\mathbf{MCISp}_{\mathbf{V}}$, we will prove that every signature morphism \mathbf{d} from a signature Σ to a signature Λ induces a $(\mathcal{U}^S)^2$ -continuous mapping from $(\text{Eq}(\Sigma), \text{Cn}_{\Sigma})$ to $(\text{Eq}(\Lambda)_{(\prod_{\varphi})^2}, (\text{Cn}_{\Lambda})_{(\prod_{\varphi})^2})$.

Also as for the single-sorted case, for every set of sorts S , there exists, up to a concrete isomorphism, a category of S -closure spaces, with objects given by an S -sorted set and, alternative, but equivalently, an S -closure system or an S -closure operator on it.

Proposition 21. *Let S be a set of sorts. Then we have that*

- (1) *The S -closure system spaces together with the S -continuous mappings between them, as defined in the first part of Definition 15, constitute a category $\mathbf{CISySp}(S)$. Furthermore, from $\mathbf{CISySp}(S)$ to \mathbf{Set}^S the forgetful functor, which sends an S -continuous mapping f from (A, \mathcal{C}) to (B, \mathcal{D}) to the S -sorted mapping f from A to B , is faithful. Therefore $\mathbf{CISySp}(S)$ is a concrete category on \mathbf{Set}^S .*
- (2) *The S -closure operator spaces together with the S -continuous mappings between them, as defined in the second part of Definition 15, constitute a category $\mathbf{CLOpSp}(S)$. Furthermore, from $\mathbf{CLOpSp}(S)$ to \mathbf{Set}^S the forgetful functor, which sends an S -continuous mapping f from (A, J) to (B, K) to the S -sorted mapping f from A to B , is faithful. Therefore $\mathbf{CLOpSp}(S)$ is a concrete category on \mathbf{Set}^S .*
- (3) *The categories $\mathbf{CLOpSp}(S)$ and $\mathbf{CISySp}(S)$ are concretely isomorphic, through the functor which sends the S -continuous mapping f from (A, J) to (B, K) to the S -continuous mapping f from $(A, \text{Fix}(J))$ to $(B, \text{Fix}(K))$.*

From now on, and for a set of sorts S , by the category of S -closure spaces, denoted by $\mathbf{CISp}(S)$, we will refer, indistinctly, to anyone of the categories $\mathbf{CISySp}(S)$ or $\mathbf{CLOpSp}(S)$.

We point out that, once more, as for the single-sorted case, for every set of sorts S , the forgetful functor from the category $\mathbf{CISp}(S)$ to the category \mathbf{Set}^S has left and right adjoints and constructs limits and colimits.

After associating to a set of sorts S the category $\mathbf{CISp}(S)$ of S -closure spaces, we prove next that a mapping $\varphi: S \longrightarrow T$ induces an adjunction $\prod_{\varphi}^{\text{cl}} \dashv \Delta_{\varphi}^{\text{cl}}$ from $\mathbf{CISp}(S)$ to $\mathbf{CISp}(T)$.

For the proof of the above assertion it will be shown to be useful to introduce the following notational conventions. Let $\varphi: S \longrightarrow T$ be a mapping and $\prod_{\varphi} \dashv \Delta_{\varphi}$ the adjunction from \mathbf{Set}^S to \mathbf{Set}^T induced by φ , then

- (1) For a T -sorted set B and a subset \mathcal{D} of $\text{Sub}(B)$, $\Delta_\varphi[\mathcal{D}]$ denotes the subset $\{D_\varphi \mid D \in \mathcal{D}\}$ of $\text{Sub}(B_\varphi)$, and
- (2) For an S -sorted set A and a subset \mathcal{C} of $\text{Sub}(A)$, $\coprod_\varphi[\mathcal{C}]$ denotes the subset $\{\coprod_\varphi C \mid C \in \mathcal{C}\}$ of $\text{Sub}(\coprod_\varphi A)$.

Proposition 22. *Let $\varphi: S \longrightarrow T$ be a mapping. Then from $\mathbf{ClSp}(T)$ to $\mathbf{ClSp}(S)$ there exists a functor $\Delta_\varphi^{\text{cl}}$ defined as follows*

- (1) $\Delta_\varphi^{\text{cl}}$ sends (B, \mathcal{D}) in $\mathbf{ClSp}(T)$ to $(B_\varphi, \Delta_\varphi[\mathcal{D}])$ in $\mathbf{ClSp}(S)$.
- (2) $\Delta_\varphi^{\text{cl}}$ sends a T -continuous mapping $f: (B, \mathcal{D}) \longrightarrow (B', \mathcal{D}')$ to the S -continuous mapping $f_\varphi: (B_\varphi, \Delta_\varphi[\mathcal{D}]) \longrightarrow (B'_\varphi, \Delta_\varphi[\mathcal{D}'])$.

Proof. Let \mathcal{D} be a T -closure system on B , then $\Delta_\varphi[\mathcal{D}]$ is an S -closure system on B_φ , because, for every family $(Y^i)_{i \in I}$ of T -sorted sets, we have that

$$(\bigcap_{i \in I} Y^i)_\varphi = \bigcap_{i \in I} Y^i_\varphi.$$

Besides, if $f: (B, \mathcal{D}) \longrightarrow (B', \mathcal{D}')$ is a T -continuous mapping and $Y'_\varphi \in \Delta_\varphi[\mathcal{D}']$, then $Y' \in \mathcal{D}'$ and $f^{-1}[Y'] \in \mathcal{D}$, thus $\Delta_\varphi(f^{-1}[Y']) \in \Delta_\varphi[\mathcal{D}]$. But $\Delta_\varphi(f^{-1}[Y'])$ is identical to $\Delta_\varphi(f)^{-1}[Y'_\varphi]$, therefore f_φ is an S -continuous mapping. \square

Proposition 23. *Let $\varphi: S \longrightarrow T$ be a mapping. Then from $\mathbf{ClSp}(S)$ to $\mathbf{ClSp}(T)$ there exists a functor $\coprod_\varphi^{\text{cl}}$ defined as follows*

- (1) $\coprod_\varphi^{\text{cl}}$ sends (A, \mathcal{C}) in $\mathbf{ClSp}(S)$ to $(\coprod_\varphi A, \coprod_\varphi[\mathcal{C}])$ in $\mathbf{ClSp}(T)$.
- (2) $\coprod_\varphi^{\text{cl}}$ sends an S -continuous mapping $f: (A, \mathcal{C}) \longrightarrow (A', \mathcal{C}')$ to the T -continuous mapping $\coprod_\varphi f: (\coprod_\varphi A, \coprod_\varphi[\mathcal{C}]) \longrightarrow (\coprod_\varphi A', \coprod_\varphi[\mathcal{C}'])$.

Proof. Let \mathcal{C} be an S -closure system on A , then $\coprod_\varphi[\mathcal{C}]$ is a T -closure system on $\coprod_\varphi A$, because, for every family $(X^i)_{i \in I}$ of S -sorted sets, we have that

$$\coprod_\varphi \bigcap_{i \in I} X^i = \bigcap_{i \in I} \coprod_\varphi X^i.$$

Besides, if $f: (A, \mathcal{C}) \longrightarrow (A', \mathcal{C}')$ is an S -continuous mapping and $\coprod_\varphi X' \in \coprod_\varphi[\mathcal{C}']$, then X' belongs to \mathcal{C}' and $f^{-1}[X'] \in \mathcal{C}$, thus $\coprod_\varphi f^{-1}[X'] \in \coprod_\varphi[\mathcal{C}]$. But $\coprod_\varphi f^{-1}[X']$ is identical to $(\coprod_\varphi f)^{-1}[\coprod_\varphi X']$, therefore $\coprod_\varphi f$ is a T -continuous mapping. \square

Proposition 24. *Let $\varphi: S \longrightarrow T$ be a mapping. Then the functor $\coprod_\varphi^{\text{cl}}$ is left adjoint to the functor $\Delta_\varphi^{\text{cl}}$.*

Proof. The natural isomorphism θ^φ of the adjunction $\coprod_\varphi \dashv \Delta_\varphi$ also happens to be a natural isomorphism

$$\text{Hom}((A, \mathcal{C}), (B_\varphi, \Delta_\varphi[\mathcal{D}])) \cong \text{Hom}((\coprod_\varphi A, \coprod_\varphi[\mathcal{C}]), (B, \mathcal{D})),$$

for every (A, \mathcal{C}) in $\mathbf{ClSp}(S)$ and every (B, \mathcal{D}) in $\mathbf{ClSp}(T)$.

Let f be an S -continuous mapping from (A, \mathcal{C}) to $(B_\varphi, \Delta_\varphi[\mathcal{D}])$, and $Y \in \mathcal{D}$. Since $Y_\varphi \in \Delta_\varphi[\mathcal{D}]$ and f is continuous, we have that $f^{-1}[Y_\varphi] \in \mathcal{C}$ and $\coprod_\varphi f^{-1}[Y_\varphi] \in \coprod_\varphi[\mathcal{C}]$. But $\coprod_\varphi f^{-1}[Y_\varphi]$ is identical to $((\theta^\varphi)^{-1}(f))^{-1}[Y]$, because

$$\begin{aligned} ((\theta^\varphi)^{-1}(f))^{-1}[Y] &= (\{(a, s) \in \coprod_\varphi(A)_t \mid a \in A_s, \varphi(s) = t, f_s(a) \in Y_t\})_{t \in T} \\ &= (\{(a, s) \in \coprod_\varphi(A)_t \mid a \in f^{-1}[Y_\varphi]_s, \varphi(s) = t\})_{t \in T} \\ &= (\coprod_{s \in \varphi^{-1}[t]} f^{-1}[Y_\varphi]_s)_{t \in T} \\ &= \coprod_\varphi f^{-1}[Y_\varphi], \end{aligned}$$

therefore $(\theta^\varphi)^{-1}(f)$ is a T -continuous mapping.

Reciprocally, let us suppose that g is a T -continuous mapping from $(\coprod_{\varphi} A, \coprod_{\varphi} [\mathcal{C}])$ to (B, \mathcal{D}) . Let $Y_{\varphi} \in \Delta_{\varphi}[\mathcal{D}]$ be, then $Y \in \mathcal{D}$ and $g^{-1}[Y] \in \coprod_{\varphi} [\mathcal{C}]$. But we have that

$$\begin{aligned} g^{-1}[Y] &= (\{(a, s) \in \coprod_{\varphi} (A)_t \mid g_t(a, s) \in Y_t\})_{t \in T} \\ &= (\coprod_{s \in \varphi^{-1}[t]} \{a \in A_s \mid g_{\varphi(s)}(a, s) \in Y_{\varphi(s)}\})_{t \in T} \\ &= \coprod_{\varphi} (\{a \in A_s \mid g_{\varphi(s)}(a, s) \in Y_{\varphi(s)}\})_{s \in S}, \end{aligned}$$

and, additionally,

$$\begin{aligned} (\{a \in A_s \mid g_{\varphi(s)}(a, s) \in Y_{\varphi(s)}\})_{s \in S} &= (\{a \in A_s \mid \theta^{\varphi}(g)_s(a) \in Y_{\varphi(s)}\})_{s \in S} \\ &= (\theta^{\varphi}(g))^{-1}[Y_{\varphi}], \end{aligned}$$

thus $g^{-1}[Y] = \coprod_{\varphi} (\theta^{\varphi}(g))^{-1}[Y_{\varphi}]$, therefore $(\theta^{\varphi}(g))^{-1}[Y_{\varphi}] \in \mathcal{C}$ and $\theta^{\varphi}(g)$ is an S -continuous mapping. \square

The functors $\Delta_{\varphi}^{\text{cl}}$ and $\coprod_{\varphi}^{\text{cl}}$ can, obviously, also be defined for S -closure operators. Actually, the definition for the functor $\coprod_{\varphi}^{\text{cl}}$, as shown in the following proposition, is immediate.

Proposition 25. *Given a mapping $\varphi: S \longrightarrow T$ and an S -closure space (A, J) , the pair $(\coprod_{\varphi} A, J_{\varphi})$ is a T -closure space, where the operator J_{φ} on $\coprod_{\varphi} A$ assigns to $\coprod_{\varphi} X$, for $X \subseteq A$, the T -sorted set $\coprod_{\varphi} J(X)$.*

Proof. The definition of the operator J_{φ} is sound, because $\text{Sub}(A) \cong \text{Sub}(\coprod_{\varphi} A) = \coprod_{\varphi} [\text{Sub}(A)]$. \square

However, the corresponding definition for the functor $\Delta_{\varphi}^{\text{cl}}$ is more involved, because for a T -sorted set B , we only have, in general, that $\Delta_{\varphi}[\text{Sub}(B)] \subseteq \text{Sub}(B_{\varphi})$.

Proposition 26. *Given a mapping $\varphi: S \longrightarrow T$ and a T -closure space (B, K) , the pair $(B_{\varphi}, K_{\varphi})$ is an S -closure space, where the operator K_{φ} on B_{φ} is defined as follows*

$$K_{\varphi} \begin{cases} \text{Sub}(B_{\varphi}) & \longrightarrow \text{Sub}(B_{\varphi}) \\ Y & \longmapsto K((\bigcup_{s \in \varphi^{-1}[t]} Y_s)_{t \in T})_{\varphi} \end{cases}$$

Proof. The definition of the operator K_{φ} as the composition of the mappings in the diagram

$$\begin{array}{ccc} \text{Sub}(B_{\varphi}) & \xrightarrow{K_{\varphi}} & \text{Sub}(B_{\varphi}) \\ \downarrow \bigcup_{\varphi, B} & & \uparrow \Delta_{\varphi, B} \\ \text{Sub}(B) & \xrightarrow{K} & \text{Sub}(B) \end{array}$$

is sound, because the mapping $\bigcup_{\varphi, B}$ from $\text{Sub}(B_{\varphi})$ to $\text{Sub}(B)$, which sends a subset Y of $\text{Sub}(B_{\varphi})$ to the subset $(\bigcup_{s \in \varphi^{-1}[t]} Y_s)_{t \in T}$ of B , is isotone and has, precisely, as right adjoint, to the mapping $\Delta_{\varphi, B}$ from $\text{Sub}(B)$ to $\text{Sub}(B_{\varphi})$, which sends a subset X of B to the subset X_{φ} of B_{φ} . \square

For a mapping $\varphi: S \longrightarrow T$, the functors $\Delta_{\varphi}^{\text{cl}}$, from $\mathbf{CISp}(T)$ to $\mathbf{CISp}(S)$, and $\coprod_{\varphi}^{\text{cl}}$, from $\mathbf{CISp}(S)$ to $\mathbf{CISp}(T)$, are the components, respectively, of a contravariant functor and of a pseudo-functor, from \mathbf{Set} to \mathbf{Cat} . In particular, by applying the construction of Ehresmann-Grothendieck to the contravariant functor we will get a category of many-sorted closure spaces.

Proposition 27. *There exists a contravariant functor Δ^{cl} from \mathbf{Set} to \mathbf{Cat} which sends a set S to $\Delta^{\text{cl}}(S) = \mathbf{ClSp}(S)$, the category of S -closure spaces, and a mapping $\varphi: S \rightarrow T$ to the functor $\Delta_{\varphi}^{\text{cl}}: \mathbf{ClSp}(T) \rightarrow \mathbf{ClSp}(S)$ defined as follows*

- (1) $\Delta_{\varphi}^{\text{cl}}$ assigns to a T -closure space (B, \mathcal{D}) the S -closure space $(B_{\varphi}, \Delta_{\varphi}[\mathcal{D}])$.
- (2) $\Delta_{\varphi}^{\text{cl}}$ assigns to a T -continuous mapping f from (B, \mathcal{D}) to (B', \mathcal{D}') the S -continuous mapping f_{φ} from $(B_{\varphi}, \Delta_{\varphi}[\mathcal{D}])$ to $(B'_{\varphi}, \Delta_{\varphi}[\mathcal{D}'])$.

Definition 16. The category \mathbf{MClSp} of many-sorted closure spaces and continuous mappings, obtained by applying the construction of Ehresmann-Grothendieck to the contravariant functor Δ^{cl} from \mathbf{Set} to \mathbf{Cat} , is $\mathbf{MClSp} = \int^{\mathbf{Set}} \Delta^{\text{cl}}$.

Therefore \mathbf{MClSp} has as objects the triples (S, A, \mathcal{C}) , where S is a set and (A, \mathcal{C}) an S -closure space, and as morphisms from (S, A, \mathcal{C}) to (T, B, \mathcal{D}) the triples $((S, A, \mathcal{C}), (\varphi, f), (T, B, \mathcal{D}))$, abbreviated to $(\varphi, f): (S, A, \mathcal{C}) \rightarrow (T, B, \mathcal{D})$, where (φ, f) is such that $\varphi: S \rightarrow T$ is a mapping and $f: (A, \mathcal{C}) \rightarrow (B_{\varphi}, \Delta_{\varphi}[\mathcal{D}])$ an S -continuous mapping. From now on, to shorten terminology, we will say *closure space* and *continuous mapping*, instead of *many-sorted closure space* and *many-sorted continuous mapping*, respectively, when this is unlikely to cause confusion.

The forgetful functor from the category \mathbf{MClSp} to the category \mathbf{MSet} has left and right adjoints and constructs limits and colimits, exactly as for the forgetful functor from the category $\mathbf{ClSp}(S)$ to the category \mathbf{Set}^S . These results follow from the following two lemmas.

Lemma 4. *Let (S, A) be a ms-set, $(S_i, A^i, \mathcal{C}^i)_{i \in I}$ a family of closure spaces and $(\varphi_I, f^I) = (\varphi_i, f^i)_{i \in I}$ a family of ms-mappings, where, for every $i \in I$, (φ_i, f^i) is a ms-mapping from (S, A) to (S_i, A^i) , i.e., φ_i is a mapping from S to S_i and $f^i = (f^i_s)_{s \in S}$ an S -sorted mapping from A to $A^i_{\varphi_i} = (A^i_{\varphi_i(s)})_{s \in S}$. Then there exists a uniquely determined closure system \mathcal{C} on A , denoted by $L^{(\varphi_I, f^I)}(S_i, A^i, \mathcal{C}^i)_{i \in I}$ and called the optimal lift of $(S_i, A^i, \mathcal{C}^i)_{i \in I}$ through (φ_I, f^I) , such that:*

- (1) For every $i \in I$, $(\varphi_i, f^i): (S, A, L^{(\varphi_I, f^I)}(S_i, A^i, \mathcal{C}^i)_{i \in I}) \rightarrow (S_i, A^i, \mathcal{C}^i)$ is a continuous mapping.
- (2) For every closure space (T, B, \mathcal{D}) and every ms-mapping (ψ, g) from (T, B) to (S, A) , if, for every $i \in I$, $(\varphi_i, f^i) \circ (\psi, g)$ is a continuous mapping from (T, B, \mathcal{D}) to $(S_i, A^i, \mathcal{C}^i)$, then (ψ, g) is a continuous mapping from (T, B, \mathcal{D}) to $(S, A, L^{(\varphi_I, f^I)}(S_i, A^i, \mathcal{C}^i)_{i \in I})$.

Besides, we have that:

- (1) For every closure system \mathcal{C} on A :

$$L^{(\text{id}_S, \text{id}_A)}(S, A, \mathcal{C}) = \mathcal{C}.$$

- (2) If, for every $i \in I$, $(S_{i,m}, A^{i,m}, \mathcal{C}^{i,m})_{m \in M_i}$ is a family of closure spaces, $(\varphi_{i,M_i}, g^{i,M_i}) = (\varphi_{i,m}, g^{i,m})_{m \in M_i}$ a family of ms-mappings, where, for every $m \in M_i$, $(\varphi_{i,m}, g^{i,m})$ is a ms-mapping from (S_i, A^i) to $(S_{i,m}, A^{i,m})$ and $\mathcal{C}^i = L^{(\varphi_{i,M_i}, g^{i,M_i})}(S_{i,m}, A^{i,m}, \mathcal{C}^{i,m})_{m \in M_i}$, then

$$L^{((\varphi_{i,M_i}, g^{i,M_i}) \circ (\varphi_I, f^I))_{i \in I}}(S_{i,m}, A^{i,m}, \mathcal{C}^{i,m})_{(i,m) \in \coprod_{i \in I} M_i} = L^{(\varphi_I, f^I)}(S_i, A^i, \mathcal{C}^i)_{i \in I}.$$

Proof. To show that there exists the optimal lift of $(S_i, A^i, \mathcal{C}^i)_{i \in I}$ through the family (φ_I, f^I) of ms-mappings, it is enough to take as $L^{(\varphi_I, f^I)}(S_i, A^i, \mathcal{C}^i)_{i \in I}$ the closure system on A generated by $\bigcup_{i \in I} \{(f^i)^{-1}[C] \mid C \in \Delta_{\varphi_i}[\mathcal{C}^i]\}$.

The remaining parts are an obvious consequence of the first part. \square

Lemma 5. *Let (S, A) be ms-set, $(S_i, A^i, \mathcal{C}^i)_{i \in I}$ a family of closure spaces and $(\varphi_I, f^I) = (\varphi_i, f^i)_{i \in I}$ a family of ms-mappings, where, for every $i \in I$, (φ_i, f^i)*

is a ms-mapping from (S_i, A^i) to (S, A) , i.e., φ_i is a mapping from S_i to S and $f^i = (f_s^i)_{s \in S_i}$ an S_i -sorted mapping from A^i to $A_{\varphi_i} = (A_{\varphi_i(s)})_{s \in S_i}$. Then there exists a uniquely determined closure system \mathcal{C} on A , denoted by $L_{(\varphi_I, f^I)}(S_i, A^i, \mathcal{C}^i)_{i \in I}$ and called the co-optimal lift of $(S_i, A^i, \mathcal{C}^i)_{i \in I}$ through (φ_I, f^I) , such that:

- (1) For every $i \in I$, $(\varphi_i, f^i): (S_i, A^i, \mathcal{C}^i) \longrightarrow (S, A, L_{(\varphi_I, f^I)}(S_i, A^i, \mathcal{C}^i)_{i \in I})$ is a continuous mapping.
- (2) For every closure space (T, B, \mathcal{D}) and every ms-mapping (ψ, g) from (S, A) to (T, B) , if, for every $i \in I$, $(\psi, g) \circ (\varphi_i, f^i)$ is a continuous mapping from $(S_i, A^i, \mathcal{C}^i)$ to (T, B, \mathcal{D}) , then (ψ, g) is a continuous mapping from $(S, A, L_{(\varphi_I, f^I)}(S_i, A^i, \mathcal{C}^i)_{i \in I})$ to (T, B, \mathcal{D}) .

Besides, we have that:

- (1) For every closure system \mathcal{C} on A :

$$L_{(\text{id}_S, \text{id}_A)}(S, A, \mathcal{C}) = \mathcal{C}.$$

- (2) If, for every $i \in I$, $(S_{i,m}, A^{i,m}, \mathcal{C}^{i,m})_{m \in M_i}$ is a family of closure spaces, $(\varphi_{i,M_i}, g^{i,M_i}) = (\varphi_{i,m}, g^{i,m})_{m \in M_i}$ a family of ms-mappings, where, for every $m \in M_i$, $(\varphi_{i,m}, g^{i,m})$ is a ms-mapping from $(S_{i,m}, A^{i,m})$ to (S_i, A^i) and $\mathcal{C}^i = L_{(\varphi_{i,M_i}, g^{i,M_i})}(S_{i,m}, A^{i,m}, \mathcal{C}^{i,m})_{m \in M_i}$, then

$$L_{((\varphi_I, f^I) \circ (\varphi_{i,M_i}, g^{i,M_i}))_{i \in I}}(S_{i,m}, A^{i,m}, \mathcal{C}^{i,m})_{(i,m) \in \coprod_{i \in I} M_i} = L_{(\varphi_I, f^I)}(S_i, A^i, \mathcal{C}^i)_{i \in I}.$$

Proof. To show that there exists the co-optimal lift of $(S_i, A^i, \mathcal{C}^i)_{i \in I}$ through the family (φ_I, f^I) of ms-mappings, let Λ be the set of all closure systems \mathcal{L} on A such that, for every $i \in I$, (φ_i, f^i) is a continuous mapping from $(S_i, A^i, \mathcal{C}^i)$ to (S, A, \mathcal{L}) . The set Λ is nonempty since the empty optimal lift is in Λ . Next, let \mathcal{C} be the optimal lift of the Λ -indexed family $(\text{id}_S, \text{id}_A): (S, A) \longrightarrow (S, A, \mathcal{L})$, (T, B, \mathcal{D}) a closure space and (ψ, g) a ms-mapping from (S, A) to (T, B) such that, for every $i \in I$, $(\psi, g) \circ (\varphi_i, f^i)$ is a continuous mapping from $(S_i, A^i, \mathcal{C}^i)$ to (T, B, \mathcal{D}) . Let \mathcal{L} be the optimal lift of (T, B, \mathcal{D}) through the ms-mapping (ψ, g) . Since \mathcal{L} is optimal, $\mathcal{L} \in \Lambda$ and (ψ, g) is a continuous mapping from (S, A, \mathcal{C}) to (T, B, \mathcal{D}) because it is the composition of $(\text{id}_S, \text{id}_A)$ and (ψ, g) .

The remaining parts are an obvious consequence of the first part. \square

From Lemmas 4 & 5 it follows, as announced above, immediately the following

Corollary 8. *The forgetful functor from \mathbf{MCISp} to \mathbf{MSet} has left and right adjoints and constructs limits and colimits.*

We observe that the properties of the category \mathbf{MCISp} could, eventually, be useful in order to facilitate the construction of logical systems dealing simultaneously with objects of two, or more, types, under the hypothesis of the existence of some kind of interaction between them (reflected at the model-theoretical level by the existence of, e.g., adjoint situations).

Remark. All of the results stated by Feitosa and D'Ottaviano in [19] (compare with those stated a long time ago by Brown in [9], by Brown and Suszko in [10], and by Porte in [51], especially those in Chapter 12, pp. 83–96) that have to do with closure spaces, continuous mappings, optimal and co-optimal lifts, and completeness and co-completeness of the category of closure spaces, fall, as a very particular case, under the just developed theory, because what they call *logics* are, simply, ordinary (not many-sorted) closure spaces. Besides, by defining the appropriate subcategories of \mathbf{MCISp} , the many-sorted counterparts of the remaining results in [19] are also, easily, provable from the above generalized theory about many-sorted closure spaces.

To make the family $(\text{Cn}_\Sigma)_{\Sigma \in \mathbf{Sig}}$ of closure operators the components of a pseudo-functor Cn from \mathbf{Sig} to a convenient 2-category of monads, we begin by proving, for a Grothendieck universe \mathcal{V} such that $\mathcal{U} \in \mathcal{V}$, the existence of two mappings from the sets of objects and morphisms of \mathbf{Sig} to the respective sets of objects and morphisms of the category $\mathbf{MCISp}_{\mathcal{V}}$, from which we will get the pseudo-functor Cn .

Proposition 28. *Let \mathcal{V} be a Grothendieck universe such that $\mathcal{U} \in \mathcal{V}$. Then there exists a pair of mappings, both denoted by Cn , one from the set of objects of \mathbf{Sig} to the set of objects of the category $\mathbf{MCISp}_{\mathcal{V}}$, of closure spaces for \mathcal{V} , which sends Σ to the closure space $\text{Cn}(\Sigma) = ((\mathcal{U}^S)^2, \text{Eq}(\Sigma), \text{Cn}_\Sigma)$, and the other from the set of morphisms of \mathbf{Sig} to the set of morphism of $\mathbf{MCISp}_{\mathcal{V}}$, which sends $\mathbf{d}: \Sigma \rightarrow \Lambda$ to the continuous mapping*

$$\text{Cn}(\mathbf{d}) = ((\coprod_{\varphi}^2)^2, \mathbf{d}_{\diamond}^2): ((\mathcal{U}^S)^2, \text{Eq}(\Sigma), \text{Cn}_\Sigma) \rightarrow ((\mathcal{U}^T)^2, \text{Eq}(\Lambda), \text{Cn}_\Lambda).$$

Proof. We restrict ourselves to prove that \mathbf{d}_{\diamond}^2 is a $(\mathcal{U}^S)^2$ -continuous mapping from $(\text{Eq}(\Sigma), \text{Cn}_\Sigma)$ to $(\text{Eq}(\Lambda)_{(\coprod_{\varphi}^2)^2}, (\text{Cn}_\Lambda)_{(\coprod_{\varphi}^2)^2})$, i.e., that, for every $\mathcal{E} \subseteq \text{Eq}(\Sigma)$, $X, Y \in \mathcal{U}^S$, and $(P, Q) \in \text{Eq}(\Sigma)_{X, Y}$, we have that

$$(P, Q) \in \text{Cn}_\Sigma(\mathcal{E})_{X, Y} \quad \text{only if} \quad (\mathbf{d}_{\diamond}(P), \mathbf{d}_{\diamond}(Q)) \in \text{Cn}_\Lambda(\mathbf{d}_{\diamond}^2[\mathcal{E}])_{\coprod_{\varphi} X, \coprod_{\varphi} Y}.$$

If $(P, Q) \in \text{Cn}_\Sigma(\mathcal{E})_{X, Y}$, then $(\mathbf{d}_{\diamond}(P), \mathbf{d}_{\diamond}(Q)) \in \text{Cn}_\Lambda(\mathbf{d}_{\diamond}^2[\mathcal{E}])_{\coprod_{\varphi} X, \coprod_{\varphi} Y}$, because, for every Λ -algebra \mathbf{A} , from $\mathbf{A} \models^{\Lambda} \mathbf{d}_{\diamond}^2[\mathcal{E}]$, by Lemma 3, it follows that $\mathbf{d}^*(\mathbf{A}) \models^{\Sigma} \mathcal{E}$, hence $\mathbf{d}^*(\mathbf{A}) \models_{X, Y}^{\Sigma} (P, Q)$, therefore $\mathbf{A} \models_{\coprod_{\varphi} X, \coprod_{\varphi} Y}^{\Lambda} (\mathbf{d}_{\diamond}(P), \mathbf{d}_{\diamond}(Q))$. \square

But Cn does not determine a functor from \mathbf{Sig} to $\mathbf{MCISp}_{\mathcal{V}}$, because, for example, for two composable morphisms $\mathbf{d}: \Sigma \rightarrow \Lambda$ and $\mathbf{e}: \Lambda \rightarrow \Omega$, it is not true, generally, that $\text{Cn}(\mathbf{e} \circ \mathbf{d}) = \text{Cn}(\mathbf{e}) \circ \text{Cn}(\mathbf{d})$. However, by defining the adequate 1-cells and 2-cells, we will get a 2-category $\mathbf{Mnd}_{\mathcal{V}, \text{alg}}$, defined below, that will act as the target 2-category for a pseudo-functor defined on \mathbf{Sig} and itself obtained from Cn .

Since the target 2-category we want to determine, $\mathbf{Mnd}_{\mathcal{V}, \text{alg}}$, will be obtained from the similar 2-category $\mathbf{Mnd}_{\text{alg}}$, simply, by changing the Grothendieck universe from \mathcal{U} to \mathcal{V} , we proceed next to define this last 2-category.

We begin by recalling the concept of adjoint square and one of the fundamental facts about it, i.e., that the adjoint squares are endowed with a structure of double category (more details about this subject matter can be found in [29], [44], and [50]), since to define $\mathbf{Mnd}_{\text{alg}}$ it will be required.

Definition 17. (Cf., [29], pp. 144–145) An *adjoint square* is a triple

$$(F \dashv G, (J, \lambda, H), F' \dashv G'),$$

where the adjoints $F \dashv G$ and $F' \dashv G'$ and the functors J and H are related as in the diagram

$$\begin{array}{ccc} \mathbf{C} & \begin{array}{c} \xleftarrow{G} \\ \top \\ \xrightarrow{F} \end{array} & \mathbf{D} \\ J \downarrow & & \downarrow H \\ \mathbf{C}' & \begin{array}{c} \xleftarrow{G'} \\ \top \\ \xrightarrow{F'} \end{array} & \mathbf{D}' \end{array}$$

and λ is a matrix

$$\lambda = \begin{pmatrix} \lambda_0: F'J \Longrightarrow HF & \lambda_1: J \Longrightarrow G'HF \\ \lambda_2: F'JG \Longrightarrow H & \lambda_3: JG \Longrightarrow G'H \end{pmatrix}$$

of compatible 2-cells, i.e., a matrix of natural transformations as indicated such that

$$\begin{aligned}\lambda_0 &= (\lambda_2 F)(F' J \eta) = (\varepsilon' H F)(F' \lambda_1) = (\varepsilon' H F)(F' \lambda_3 F)(F' J \eta), \\ \lambda_1 &= (G' \lambda_0)(\eta' J) = (G' \lambda_2 F)(\eta' J \eta) = (\lambda_3 F)(J \eta), \\ \lambda_2 &= (H \varepsilon)(\lambda_0 G) = (\varepsilon' H \varepsilon)(F' \lambda_1 G) = (\varepsilon' H)(F' \lambda_3), \\ \lambda_3 &= (G' H \varepsilon)(G' \lambda_0 G)(\eta' J G) = (G' \lambda_2)(\eta' J G) = (G' H \varepsilon)(\lambda_1 G),\end{aligned}$$

where $\eta: 1 \Longrightarrow GF$ and $\varepsilon: FG \Longrightarrow 1$ are the unit and counit of $F \dashv G$, and $\eta': 1 \Longrightarrow G'F'$ and $\varepsilon': F'G' \Longrightarrow 1$ the unit and counit of $F' \dashv G'$.

In the following proposition it is stated that the adjoint squares form a double category. We do not give a proof of this proposition, since one by Gray can be found in [29], pp. 146–149.

However, following the proposition we will give explicit details about the defining data of the double category of adjoint squares, to obviate the search in the original sources ([29], [44], and [50]), and because some of them will be needful below (when defining the algebraic morphisms between monads and the algebraic transformations from an algebraic morphism into a like one).

Proposition 29. *The adjoint squares constitute a double category, denoted by **AdFun**.*

As announced above, we recall the definition of the data that occur in the double category **AdFun**.

Given an adjoint square $(F \dashv G, (J, \lambda, H), F' \dashv G')$ its *Ad-domain* and *Ad-codomain* in **AdFun** are $F \dashv G$ and $F' \dashv G'$, respectively, and its *Fun-domain* and *Fun-codomain* in **AdFun** are J and H , respectively.

The *Ad-identities* and *Fun-identities* are represented by the following adjoint squares

$$\begin{array}{ccc} \mathbf{C} & \xleftarrow{1} & \mathbf{C} \\ \downarrow J & \begin{array}{c} \xrightarrow{1} \\ \left(\begin{array}{cc} J & J \\ J & J \end{array} \right) \\ \xrightarrow{1} \end{array} & \downarrow J \\ \mathbf{C}' & \xleftarrow{1} & \mathbf{C}' \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{C} & \xleftarrow{G} & \mathbf{D} \\ \downarrow 1 & \begin{array}{c} \xrightarrow{F} \\ \left(\begin{array}{cc} F & \eta \\ \varepsilon & G \end{array} \right) \\ \xrightarrow{G} \end{array} & \downarrow 1 \\ \mathbf{C} & \xleftarrow{G} & \mathbf{D} \end{array}$$

The *Ad-composition* of two adjoint squares

$$\begin{array}{ccccc} \mathbf{C} & \xleftarrow{G} & \mathbf{D} & \xleftarrow{R} & \mathbf{E} \\ \downarrow J & \begin{array}{c} \xrightarrow{F} \\ \lambda \\ \xrightarrow{F} \end{array} & \downarrow H & \begin{array}{c} \xrightarrow{L} \\ \delta \\ \xrightarrow{L} \end{array} & \downarrow M \\ \mathbf{C}' & \xleftarrow{G'} & \mathbf{D}' & \xleftarrow{R'} & \mathbf{E}' \\ & \xrightarrow{F'} & & \xrightarrow{L'} & \end{array}$$

is the adjoint square $(LF \dashv GR, (J, \delta \circ^{\text{ad}} \lambda, M), L'F' \dashv G'R')$ where $\delta \circ^{\text{ad}} \lambda$ is the matrix

$$\delta \circ^{\text{ad}} \lambda = \begin{pmatrix} (\delta_0 F)(L' \lambda_0) & (G' \delta_1 F) \lambda_1 \\ \delta_2 (L' \lambda_2 R) & (G' \delta_3) (\lambda_3 R) \end{pmatrix}.$$

And the Fun-*composition* of two adjoint squares

$$\begin{array}{ccc}
 \mathbf{C} & \begin{array}{c} \xleftarrow{G} \\ \xrightarrow{F} \end{array} & \mathbf{D} \\
 \downarrow J & \lambda & \downarrow H \\
 \mathbf{C}' & \begin{array}{c} \xleftarrow{G'} \\ \xrightarrow{F'} \end{array} & \mathbf{D}' \\
 \downarrow J' & \lambda' & \downarrow H' \\
 \mathbf{C}'' & \begin{array}{c} \xleftarrow{G''} \\ \xrightarrow{F''} \end{array} & \mathbf{D}''
 \end{array}$$

is the adjoint square $(F \dashv G, (J'J, \lambda' \overset{\text{fn}}{\circ} \lambda, H'H), F'' \dashv G'')$ where $\lambda' \overset{\text{fn}}{\circ} \lambda$ is the matrix

$$\lambda' \overset{\text{fn}}{\circ} \lambda = \begin{pmatrix} (H'\lambda_0)(\lambda'_0 J) & (G'H'\varepsilon'HF)(\lambda'_1 \lambda_1) \\ (\lambda'_2 \lambda_2)(F''J'\eta'JG) & (\lambda'_3 H)(J'\lambda_3) \end{pmatrix}.$$

To attain our aim of defining the 2-category $\mathbf{Mnd}_{\text{alg}}$, we continue by defining the concept of monad and by stating, for a pair of monads and an adjunction between the underlying categories of the monads, the existence of a commutative square of bijections between four sets of natural transformations obtained from the monads and the adjunction, as well as conditions of compatibility on the matrices of natural transformations arranged in the pattern of the just named commutative square of bijections.

Definition 18. By a *monad* we understand a pair (\mathbf{C}, \mathbb{T}) , with \mathbf{C} a category and $\mathbb{T} = (T, \eta, \mu)$ a monad in \mathbf{C} .

Proposition 30. Let (\mathbf{C}, \mathbb{T}) and $(\mathbf{C}', \mathbb{T}')$ be two monads and $(J, K, \bar{\eta}, \bar{\varepsilon}): \mathbf{C} \longrightarrow \mathbf{C}'$ an adjunction. Then for the following diagram

$$\begin{array}{ccc}
 \mathbf{C} & \xrightarrow{T} & \mathbf{C} \\
 \downarrow J \dashv K & & \downarrow J \dashv K \\
 \mathbf{C}' & \xrightarrow{T'} & \mathbf{C}'
 \end{array}$$

there exists, by Corollary I,6.6 stated by Gray in [29], p. 143, a commutative square of bijections

$$\begin{array}{ccc}
 \text{Nat}(JT, T'J) & \xrightarrow{\cong} & \text{Nat}(T, KT'J) \\
 \cong \downarrow & & \downarrow \cong \\
 \text{Nat}(JTK, T') & \xrightarrow{\cong} & \text{Nat}(TK, KT')
 \end{array}$$

Furthermore, the following conditions on the natural transformations in the matrix

$$\lambda = \begin{pmatrix} \lambda_0: JT \Longrightarrow T'J & \lambda_1: T \Longrightarrow KT'J \\ \lambda_2: JTK \Longrightarrow T' & \lambda_3: TK \Longrightarrow KT' \end{pmatrix}$$

are compatible with the above bijections:

(1) The natural transformations $\lambda_0: JT \Longrightarrow T'J$ such that

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} 1 \\ \downarrow \eta \\ \dashrightarrow T \dashrightarrow \\ \downarrow J \quad \Downarrow \lambda_0 \quad \downarrow J \\ \dashrightarrow T' \dashrightarrow \end{array} & = & \begin{array}{ccc}
 \begin{array}{c} 1 \\ \dashrightarrow \\ \downarrow J \quad \Downarrow J \quad \downarrow J \\ \dashrightarrow 1 \dashrightarrow \\ \downarrow T' \end{array} & & \begin{array}{ccc}
 \begin{array}{c} T \quad T \\ \dashrightarrow \quad \dashrightarrow \\ \downarrow J \quad \Downarrow \lambda_0 \quad \downarrow J \quad \Downarrow \lambda_0 \quad \downarrow J \\ \dashrightarrow T' \dashrightarrow T' \dashrightarrow \\ \downarrow \mu' \end{array} & = & \begin{array}{ccc}
 \begin{array}{c} TT \\ \downarrow \mu \\ \dashrightarrow T \dashrightarrow \\ \downarrow J \quad \Downarrow \lambda_0 \quad \downarrow J \\ \dashrightarrow T' \dashrightarrow \end{array}
 \end{array}
 \end{array}$$

(2) The natural transformations $\lambda_1: T \Longrightarrow KT'J$ such that

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} 1 \\ \downarrow \eta \\ \dashrightarrow T \dashrightarrow \\ \downarrow J \quad \Downarrow \lambda_1 \quad \downarrow K \\ \dashrightarrow T' \dashrightarrow \end{array} & = & \begin{array}{ccc}
 \begin{array}{c} 1 \\ \dashrightarrow \\ \downarrow J \quad \Downarrow \bar{\eta} \quad \downarrow K \\ \dashrightarrow 1 \dashrightarrow \\ \downarrow T' \end{array} & & \begin{array}{ccc}
 \begin{array}{c} T \quad 1 \quad T \\ \dashrightarrow \quad \dashrightarrow \quad \dashrightarrow \\ \downarrow J \quad \Downarrow \lambda_1 \quad \downarrow K \quad \Downarrow \bar{\varepsilon} \quad \downarrow J \quad \Downarrow \lambda_1 \quad \downarrow K \\ \dashrightarrow T' \dashrightarrow 1 \dashrightarrow T' \dashrightarrow \\ \downarrow \mu' \end{array} & = & \begin{array}{ccc}
 \begin{array}{c} TT \\ \downarrow \mu \\ \dashrightarrow T \dashrightarrow \\ \downarrow J \quad \Downarrow \lambda_1 \quad \downarrow K \\ \dashrightarrow T' \dashrightarrow \end{array}
 \end{array}
 \end{array}$$

(3) The natural transformations $\lambda_2: JTK \Longrightarrow T'$ such that

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} 1 \\ \downarrow \eta \\ \dashrightarrow T \dashrightarrow \\ \uparrow K \quad \Downarrow \lambda_2 \quad \downarrow J \\ \dashrightarrow T' \dashrightarrow \end{array} & = & \begin{array}{ccc}
 \begin{array}{c} 1 \\ \dashrightarrow \\ \uparrow K \quad \Downarrow \bar{\varepsilon} \quad \downarrow J \\ \dashrightarrow 1 \dashrightarrow \\ \downarrow T' \end{array} & & \begin{array}{ccc}
 \begin{array}{c} T \quad 1 \quad T \\ \dashrightarrow \quad \dashrightarrow \quad \dashrightarrow \\ \uparrow K \quad \Downarrow \lambda_2 \quad \downarrow J \quad \Downarrow \bar{\eta} \quad \uparrow K \quad \Downarrow \lambda_2 \quad \downarrow J \\ \dashrightarrow T' \dashrightarrow 1 \dashrightarrow T' \dashrightarrow \\ \downarrow \mu' \end{array} & = & \begin{array}{ccc}
 \begin{array}{c} TT \\ \downarrow \mu \\ \dashrightarrow T \dashrightarrow \\ \uparrow K \quad \Downarrow \lambda_2 \quad \downarrow J \\ \dashrightarrow T' \dashrightarrow \end{array}
 \end{array}
 \end{array}$$

(4) The natural transformations $\lambda_3: TK \Longrightarrow KT'$ such that

$$\begin{array}{c}
 \begin{array}{ccc}
 \begin{array}{c} 1 \\ \downarrow \eta \\ \dashrightarrow T \dashrightarrow \\ \uparrow K \quad \Downarrow \lambda_3 \quad \uparrow K \\ \dashrightarrow T' \dashrightarrow \end{array} & = & \begin{array}{ccc}
 \begin{array}{c} 1 \\ \dashrightarrow \\ \uparrow K \quad \Downarrow K \quad \uparrow K \\ \dashrightarrow 1 \dashrightarrow \\ \downarrow T' \end{array} & & \begin{array}{ccc}
 \begin{array}{c} T \quad T \\ \dashrightarrow \quad \dashrightarrow \\ \uparrow K \quad \Downarrow \lambda_3 \quad \uparrow K \quad \Downarrow \lambda_3 \quad \uparrow K \\ \dashrightarrow T' \dashrightarrow T' \dashrightarrow \\ \downarrow \mu' \end{array} & = & \begin{array}{ccc}
 \begin{array}{c} TT \\ \downarrow \mu \\ \dashrightarrow T \dashrightarrow \\ \uparrow K \quad \Downarrow \lambda_3 \quad \uparrow K \\ \dashrightarrow T' \dashrightarrow \end{array}
 \end{array}
 \end{array}$$

Next we proceed to define the concept of algebraic morphism between monads, the composition of algebraic morphisms, and the identity algebraic morphisms at the monads, from which we will get the *horizontal* component of the 2-category under consideration.

Definition 19. Let (\mathbf{C}, \mathbb{T}) and $(\mathbf{C}', \mathbb{T}')$ be two monads. An *algebraic morphism*, or, to shorten terminology, an *alg-morphism* from (\mathbf{C}, \mathbb{T}) to $(\mathbf{C}', \mathbb{T}')$ is an adjoint square $(J \dashv K, (T, \lambda, T'), J \dashv K)$, also denoted by $(J \dashv K, \lambda)$, or, geometrically, by

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{T} & \mathbf{C} \\ J \downarrow \dashv \uparrow K & \lambda & J \downarrow \dashv \uparrow K \\ \mathbf{C}' & \xrightarrow{T'} & \mathbf{C}' \end{array}$$

such that the components of the matrix $\lambda = \begin{pmatrix} \lambda_0 & \lambda_1 \\ \lambda_2 & \lambda_3 \end{pmatrix}$ are compatible as in the last proposition. Then taking as objects the monads (\mathbf{C}, \mathbb{T}) such that \mathbf{C} is in \mathcal{U} , as morphisms between monads the alg-morphisms, as the identity at a monad (\mathbf{C}, \mathbb{T}) the pair $(1 \dashv 1, (\frac{T}{T} \frac{T}{T}))$, and as composition of two alg-morphisms precisely their Ad-composition as adjoint squares, we get a category, that we denote by $\mathbf{Mnd}_{\text{alg}}$.

We now define for two alg-morphisms $(J \dashv K, \lambda)$ and $(J' \dashv K', \lambda')$ from (\mathbf{C}, \mathbb{T}) to $(\mathbf{C}', \mathbb{T}')$ the concept of algebraic transformation from $(J \dashv K, \lambda)$ to $(J' \dashv K', \lambda')$ as well as the vertical and horizontal composition of algebraic transformations. This will constitute the *vertical* component of the 2-category under consideration.

Definition 20. Let

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{T} & \mathbf{C} \\ J \downarrow \dashv \uparrow K & \lambda & J \downarrow \dashv \uparrow K \\ \mathbf{C}' & \xrightarrow{T'} & \mathbf{C}' \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathbf{C} & \xrightarrow{T} & \mathbf{C} \\ J' \downarrow \dashv \uparrow K' & \lambda' & J' \downarrow \dashv \uparrow K' \\ \mathbf{C}' & \xrightarrow{T'} & \mathbf{C}' \end{array}$$

be two alg-morphisms from (\mathbf{C}, \mathbb{T}) to $(\mathbf{C}', \mathbb{T}')$. Then an *algebraic transformation*, or, to shorten terminology, an *alg-transformation*, from $(J \dashv K, \lambda)$ to $(J' \dashv K', \lambda')$ is an adjoint square

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{1} & \mathbf{C} \\ J \downarrow \dashv \uparrow K & \xi & J' \downarrow \dashv \uparrow K' \\ \mathbf{C}' & \xrightarrow{T'} & \mathbf{C}' \end{array}$$

such that

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{T} & \mathbf{C} & \xrightarrow{1} & \mathbf{C} \\ J \downarrow \dashv \uparrow K & \lambda & J \downarrow \dashv \uparrow K & \xi & J' \downarrow \dashv \uparrow K' \\ \mathbf{C}' & \xrightarrow{T'} & \mathbf{C}' & \xrightarrow{T'} & \mathbf{C}' \\ 1 \downarrow \dashv \uparrow 1 & & \mu' & & 1 \downarrow \dashv \uparrow 1 \\ \mathbf{C}' & \xrightarrow{T'} & \mathbf{C}' & & \mathbf{C}' \end{array} = \begin{array}{ccc} \mathbf{C} & \xrightarrow{1} & \mathbf{C} & \xrightarrow{T} & \mathbf{C} \\ J \downarrow \dashv \uparrow K & \xi & J' \downarrow \dashv \uparrow K' & \lambda' & J' \downarrow \dashv \uparrow K' \\ \mathbf{C}' & \xrightarrow{T'} & \mathbf{C}' & \xrightarrow{T'} & \mathbf{C}' \\ 1 \downarrow \dashv \uparrow 1 & & \mu' & & 1 \downarrow \dashv \uparrow 1 \\ \mathbf{C}' & \xrightarrow{T'} & \mathbf{C}' & & \mathbf{C}' \end{array}$$

i.e., such that $\mu' \circ^{\text{ad}} (\xi \circ^{\text{fn}} \lambda) = \mu' \circ^{\text{ad}} (\lambda' \circ^{\text{fn}} \xi)$.

For every alg-morphism $(J \dashv K, \lambda): (\mathbf{C}, \mathbb{T}) \longrightarrow (\mathbf{C}', \mathbb{T}')$, the *identity* at $(J \dashv K, \lambda)$ is the adjoint square determined by the matrix

$$\begin{pmatrix} \eta' J & K \eta' J \eta^{J \dashv K} \\ \eta' \varepsilon^{J \dashv K} & K \eta' \end{pmatrix},$$

where $\eta^{J \dashv K}$ is the unit and $\varepsilon^{J \dashv K}$ the counit of the adjunction $J \dashv K$.

The *vertical* composition of two alg-transformations as in the following diagram

$$\begin{array}{ccc} & (J \dashv K, \lambda) & \\ & \curvearrowright & \\ (\mathbf{C}, \mathbb{T}) & \xrightarrow{(J' \dashv K', \lambda')} & (\mathbf{C}', \mathbb{T}'), \\ & \curvearrowleft & \\ & (J'' \dashv K'', \lambda'') & \end{array}$$

$\downarrow \xi$
 $\downarrow \xi'$

denoted by $\xi' \circ \xi$, is the adjoint square $\mu' \circ^{\text{ad}} (\xi' \circ^{\text{fn}} \xi)$.

The *horizontal* composition of two alg-transformations as in the following diagram

$$\begin{array}{ccccc} & (J \dashv K, \lambda) & & (J'' \dashv K'', \lambda'') & \\ & \curvearrowright & & \curvearrowright & \\ (\mathbf{C}, \mathbb{T}) & \xrightarrow{\xi} & (\mathbf{C}', \mathbb{T}') & \xrightarrow{\xi'} & (\mathbf{C}'', \mathbb{T}''), \\ & \curvearrowleft & & \curvearrowleft & \\ & (J' \dashv K', \lambda') & & (J''' \dashv K''', \lambda''') & \end{array}$$

denoted by $\xi' \tilde{*} \xi$, is the adjoint square

$$\mu' \circ^{\text{ad}} (\lambda'' \circ^{\text{fn}} \xi') \circ^{\text{ad}} \xi = \mu' \circ^{\text{ad}} (\xi' \circ^{\text{fn}} \lambda''') \circ^{\text{ad}} \xi.$$

From the horizontal and vertical components just defined, it follows immediately the following

Proposition 31. *The monads whose underlying category is in \mathcal{U} , together with the alg-morphisms between monads, and the alg-transformations from an alg-morphism into a like one determine a 2-category, denoted by $\mathbf{Mnd}_{\text{alg}}$.*

Definition 21. We denote by $\mathbf{Mnd}_{\mathcal{V}, \text{alg}}$ the 2-category with objects the monads (\mathbf{C}, \mathbb{T}) such that \mathbf{C} is in \mathcal{V} , 1-cells the alg-morphisms and 2-cells the alg-transformations between alg-morphisms.

Proposition 32. *The category $\mathbf{MCISp}_{\mathcal{V}}$ can be identified to a subcategory of (the underlying category of) the 2-category $\mathbf{Mnd}_{\mathcal{V}, \text{alg}}$.*

Proof. It is enough to assign to a closure space (S, A, J) the monad $(\mathbf{Sub}(A), \mathbb{T}_J)$, where $\mathbf{Sub}(A)$ is the category determined by the ordered set $(\text{Sub}(A), \subseteq)$ and \mathbb{T}_J the monad on $\mathbf{Sub}(A)$ obtained from the S -sorted closure operator J on A , as in [42], p. 139; and to a continuous mapping (φ, f) from (S, A, J) to (T, B, K) the alg-morphism

$$\begin{array}{ccc} \mathbf{Sub}(A) & \xrightarrow{T_J} & \mathbf{Sub}(A) \\ \downarrow \cup_{\varphi, B} \circ f[\cdot] \quad \uparrow f^{-1}[\cdot] \circ \Delta_{\varphi, B} & & \downarrow \cup_{\varphi, B} \circ f[\cdot] \quad \uparrow f^{-1}[\cdot] \circ \Delta_{\varphi, B} \\ \mathbf{Sub}(B) & \xrightarrow{T_K} & \mathbf{Sub}(B) \end{array}$$

from $(\mathbf{Sub}(A), \mathbb{T}_J)$ to $(\mathbf{Sub}(B), \mathbb{T}_K)$. Observe that we have not written the matrix λ of the alg-morphism because, in this case, it is trivial. \square

Remark. Taking into account the just stated proposition, we can induce, in a derived way, for two continuous mappings (φ, f) and (ψ, g) from a closure space (S, A, J) into a like one (T, B, K) , a notion of alg-transformation from (φ, f) to (ψ, g) . But, because the underlying categories of the monads associated to the given closure spaces are complete lattices, hence preorders, there will be at most an alg-transformation from the first continuous mapping to the second one. Actually, there will be an alg-transformation from (φ, f) to (ψ, g) exactly if, for every $X \subseteq A$, we have that $\bigcup_{\psi, B} g[X] \subseteq K(\bigcup_{\varphi, B} f[X])$.

Proposition 33. *Let \mathcal{V} be a Grothendieck universe such that $\mathcal{U} \in \mathcal{V}$. Then there exists a pseudo-functor, also denoted by \mathbf{Cn} , from \mathbf{Sig} to $\mathbf{Mnd}_{\mathcal{V}, \text{alg}}$ that has as components, essentially, the consequence operators \mathbf{Cn}_{Σ} , for the different signatures Σ .*

Proof. We restrict ourselves to define the object and morphism mappings of the pseudo-functor \mathbf{Cn} . The object mapping of \mathbf{Cn} is that which sends a signature Σ to the monad $(\mathbf{Sub}(\text{Eq}(\Sigma)), \mathbb{T}_{\mathbf{Cn}_{\Sigma}})$, and the morphism mapping of \mathbf{Cn} is that which sends a signature morphism $\mathbf{d}: \Sigma \longrightarrow \Lambda$ to the alg-morphism

$$\left(\bigcup_{(\coprod_{\varphi})^2, \text{Eq}(\Lambda)} \circ \mathbf{d}_{\diamond}^2[\cdot] \dashv (\mathbf{d}_{\diamond}^2)^{-1}[\cdot] \circ \Delta_{(\coprod_{\varphi})^2, \text{Eq}(\Lambda)} \right)$$

from the monad $(\mathbf{Sub}(\text{Eq}(\Sigma)), \mathbb{T}_{\mathbf{Cn}_{\Sigma}})$ to the monad $(\mathbf{Sub}(\text{Eq}(\Lambda)), \mathbb{T}_{\mathbf{Cn}_{\Lambda}})$.

So defined, it is obvious that \mathbf{Cn} is a pseudo-functor from \mathbf{Sig} to $\mathbf{Mnd}_{\mathcal{V}, \text{alg}}$. \square

Relying on the results just stated we propose the following notion of entailment system, that generalizes that one by Meseguer in [47], pp. 282–283.

Definition 22. An *entailment system* is a quintuple $\mathcal{E} = (\mathbf{Sig}, \mathbb{T}, L, M, \mathbf{Cn})$ with \mathbf{Sig} a \mathcal{U} -category whose objects are called *signatures*, \mathbb{T} a pseudo-functor from \mathbf{Sig} to \mathbf{Cat} , L a functor from \mathbf{Cat} to $\mathbf{Set}_{\mathcal{V}}$, M a functor from $\mathbf{Set}_{\mathcal{V}}$ to $\mathbf{Set}_{\mathcal{V}}$, and \mathbf{Cn} a pseudo-functor from \mathbf{Sig} to $\mathbf{Mnd}_{\mathcal{V}, \text{alg}}$ such that the following diagram commutes up to isomorphism

$$\begin{array}{ccc} \mathbf{Sig} & \xrightarrow{\mathbf{Cn}} & \mathbf{Mnd}_{\mathcal{V}, \text{alg}} \\ \mathbb{T} \downarrow & \searrow \text{Sen} & \downarrow \pi_0 \\ \mathbf{Cat} & \xrightarrow{L} \mathbf{Set}_{\mathcal{V}} \xrightarrow{M} & \mathbf{Set}_{\mathcal{V}} \end{array}$$

where $\text{Sen} = L \circ \mathbb{T}$ and π_0 the functor from $\mathbf{Mnd}_{\mathcal{V}, \text{alg}}$ to $\mathbf{Set}_{\mathcal{V}}$ extracting the underlying set of the set of objects of the first component of its pairs.

Taking into account that, for every signature Σ , we have that $\mathbf{Sub}(\text{Eq}(\Sigma))$ and $\mathbf{Sub}(\coprod \text{Eq}(\Sigma))$ are isomorphic, we get the following

Corollary 9. *The quintuple $\mathcal{E}qcn = (\mathbf{Sig}, \text{Ter}, L, M, \mathbf{Cn})$ with \mathbf{Sig} the category of signatures, L the functor from \mathbf{Cat} to $\mathbf{Set}_{\mathcal{V}}$ which sends a \mathcal{U} -category \mathbf{C} to the \mathcal{V} -small set $\prod_{x, y \in \mathbf{C}} \text{Hom}(x, y)^2$, and M the covariant power set endofunctor of $\mathbf{Set}_{\mathcal{V}}$, is an entailment system, the so-called many-sorted equational consequence entailment system, or equational consequence entailment system.*

Observe that the many-sorted equational consequence entailment system $\mathcal{E}qcn$ embodies the essentials of the syntactical many-sorted equational deduction.

After clarifying category-theoretically the concept of equational consequence, we proceed to define the concept of many-sorted specification and that of many-sorted specification morphism.

Definition 23. A *many-sorted specification* is a pair (Σ, \mathcal{E}) , where Σ is a signature while $\mathcal{E} \subseteq \text{Eq}(\Sigma)$. A *many-sorted specification morphism* from (Σ, \mathcal{E}) to (Λ, \mathcal{H}) is a signature morphism $\mathbf{d}: \Sigma \longrightarrow \Lambda$ such that $\mathbf{d}_\diamond^2[\mathcal{E}] \subseteq \text{Cn}_\Lambda(\mathcal{H})$. From now on, to shorten terminology, we will say *specification* and *specification morphism* instead of *many-sorted specification* and *many-sorted specification morphism*, respectively. Besides, if in a specification (Σ, \mathcal{E}) the set \mathcal{E} of equations is closed, i.e., $\text{Cn}_\Sigma(\mathcal{E}) = \mathcal{E}$, then we call (Σ, \mathcal{E}) a *theory*. To abbreviate, we write, sometimes, $\bar{\mathcal{E}}$ instead of $\text{Cn}_\Sigma(\mathcal{E})$.

Proposition 34. *The specifications and the specification morphisms determine a category denoted as **Spf**.*

Proof. We restrict ourselves to prove that the composition of specification morphisms is a specification morphism.

Before proving this, let us remark that if $\mathbf{d}: \Sigma \longrightarrow \Lambda$ and $\mathbf{e}: \Lambda \longrightarrow \Omega$ are signature morphisms, (P, Q) a Σ -equation of type (X, Y) and \mathbf{C} a Ω -algebra, then $\mathbf{e}_\diamond(\mathbf{d}_\diamond(P))^\mathbf{C} = \mathbf{e}_\diamond(\mathbf{d}_\diamond(Q))^\mathbf{C}$ iff $(\mathbf{e} \circ \mathbf{d})_\diamond(P)^\mathbf{C} = (\mathbf{e} \circ \mathbf{d})_\diamond(Q)^\mathbf{C}$. Therefore, for every family of Σ -equations \mathcal{E} , we have that $\text{Cn}_\Omega(\mathbf{e}_\diamond^2[\mathbf{d}_\diamond^2[\mathcal{E}]]) = \text{Cn}_\Omega((\mathbf{e} \circ \mathbf{d})_\diamond^2[\mathcal{E}])$. Now, if $\mathbf{d}: (\Sigma, \mathcal{E}) \longrightarrow (\Lambda, \mathcal{H})$ and $\mathbf{e}: (\Lambda, \mathcal{H}) \longrightarrow (\Omega, \mathcal{F})$ are specification morphisms, then

$$\mathbf{e}_\diamond^2[\mathbf{d}_\diamond^2[\mathcal{E}]] \subseteq \mathbf{e}_\diamond^2[\text{Cn}_\Lambda(\mathcal{H})] \subseteq \text{Cn}_\Omega(\mathbf{e}_\diamond^2[\mathcal{H}]) \subseteq \text{Cn}_\Omega(\mathcal{F}),$$

from which the proposition follows. \square

Remark. The category \mathbf{Th}_b with objects the theories and morphisms from one theory to another the, so-called by Bénabou in [2], p. (sub) 27, *banal* morphisms (also known as *axiom-preserving* morphisms), is

$$\mathbf{Th}_b = \int^{\mathbf{Sig}} \text{Fix} \circ \text{Cn},$$

where Fix is the contravariant functor from $\mathbf{Mnd}_{\mathcal{V}, \text{alg}}$ to $\mathbf{Cat}_{\mathcal{V}}$ which sends a monad (\mathbf{C}, \mathbb{T}) for \mathcal{V} , to the preordered set $\mathbf{Fix}(\mathbb{T}) = (\text{Fix}(\mathbb{T}), \preceq)$, of the fixed points of \mathbb{T} , being $\text{Fix}(\mathbb{T})$ the set of all \mathbb{T} -algebras (A, δ) such that the structural morphism δ from $\mathbb{T}(A)$ to A is an isomorphism, and \preceq the preorder on $\text{Fix}(\mathbb{T})$ defined by imposing that $(A, \delta) \preceq (A', \delta')$ iff there exists a \mathbb{T} -homomorphism from (A, δ) to (A', δ') . Therefore, informally speaking, we can say that the world of theories, \mathbf{Th}_b , is the totalization over \mathbf{Sig} of the fixed points of the consequences.

We state next some, obvious, relations between the categories \mathbf{Sig} and \mathbf{Spf} that, notwithstanding, will show to be useful shortly afterwards. Every signature Σ determines the specification (Σ, \emptyset) , the so-called *indiscrete specification*, from which we get an inclusion functor

$$\text{sp}_i: \mathbf{Sig} \longrightarrow \mathbf{Spf}$$

that, in its turn, is left adjoint to the forgetful functor

$$\text{sig}: \mathbf{Spf} \longrightarrow \mathbf{Sig}$$

which sends a specification (Σ, \mathcal{E}) to the underlying signature Σ . Besides, \mathbf{Sig} is a retract of \mathbf{Spf} , i.e., $\text{sig} \circ \text{sp}_i = \text{Id}_{\mathbf{Sig}}$.

The functor sig , on its part, has a right adjoint

$$\text{sp}_d: \mathbf{Sig} \longrightarrow \mathbf{Spf}$$

which sends a signature Σ to $(\Sigma, \text{Eq}(\Sigma))$, the so-called *discrete specification*.

What we want now is to lift the contravariant functor \mathbf{Alg} from \mathbf{Sig} to \mathbf{Cat} to the category \mathbf{Spf} , by assigning, in particular, to a specification (Σ, \mathcal{E}) the category $\mathbf{Alg}(\Sigma, \mathcal{E})$ of its models.

Proposition 35. *There exists a contravariant functor $\mathbf{Alg}^{\mathbf{sp}}$ from \mathbf{Spf} to \mathbf{Cat} defined as follows*

- (1) $\mathbf{Alg}^{\mathbf{sp}}$ sends a specification (Σ, \mathcal{E}) to the category $\mathbf{Alg}^{\mathbf{sp}}(\Sigma, \mathcal{E}) = \mathbf{Alg}(\Sigma, \mathcal{E})$ of its models, i.e., the full subcategory of $\mathbf{Alg}(\Sigma)$ determined by those Σ -algebras which satisfy all the equations in \mathcal{E} .
- (2) $\mathbf{Alg}^{\mathbf{sp}}$ sends a specification morphism \mathbf{d} from (Σ, \mathcal{E}) to (Λ, \mathcal{H}) to the functor $\mathbf{Alg}^{\mathbf{sp}}(\mathbf{d}) = \mathbf{d}^*$ from $\mathbf{Alg}(\Lambda, \mathcal{H})$ to $\mathbf{Alg}(\Sigma, \mathcal{E})$, obtained from the functor \mathbf{d}^* from $\mathbf{Alg}(\Lambda)$ to $\mathbf{Alg}(\Sigma)$ by bi-restriction.

Proof. Let \mathbf{B} be a Λ -algebra such that $\mathbf{B} \models^{\Lambda} \mathcal{H}$. Then $\mathbf{B} \models^{\Lambda} \mathbf{Cn}_{\Lambda}(\mathcal{H})$, therefore $\mathbf{B} \models^{\Lambda} \mathbf{d}_{\diamond}^2[\mathcal{E}]$ hence, by Lemma 3, $\mathbf{d}^*(\mathbf{B}) \models^{\Sigma} \mathcal{E}$. \square

Remark. By applying the construction of Ehresmann-Grothendieck to the contravariant functor $\mathbf{Alg}^{\mathbf{sp}}$ from \mathbf{Spf} to \mathbf{Cat} we get the category $\mathbf{Alg}^{\mathbf{sp}} = \int^{\mathbf{Spf}} \mathbf{Alg}^{\mathbf{sp}}$ into which is embedded the category \mathbf{Alg} as a retract (because \mathbf{Sig} is a retract of \mathbf{Spf}).

On the other hand, taking care of the Completeness Theorem in [13], every family of equations $\mathcal{E} \subseteq \mathbf{Eq}(\Sigma)$ determines a congruence on the category $\mathbf{Ter}(\Sigma)$, hence a quotient category $\mathbf{Ter}(\Sigma)/\bar{\mathcal{E}}$. Besides, this procedure can be completed, as stated in the following proposition, up to a pseudo-functor $\mathbf{Ter}^{\mathbf{sp}}$ from \mathbf{Spf} to \mathbf{Cat} , and the restriction of $\mathbf{Ter}^{\mathbf{sp}}$ to \mathbf{Sig} is precisely the pseudo-functor \mathbf{Ter} .

Proposition 36. *There exists a pseudo-functor $\mathbf{Ter}^{\mathbf{sp}}$ from \mathbf{Spf} to \mathbf{Cat} defined as follows*

- (1) $\mathbf{Ter}^{\mathbf{sp}}$ sends a specification (Σ, \mathcal{E}) to the category $\mathbf{Ter}^{\mathbf{sp}}(\Sigma, \mathcal{E}) = \mathbf{Ter}(\Sigma, \mathcal{E})$, where $\mathbf{Ter}(\Sigma, \mathcal{E})$ is the quotient category $\mathbf{Ter}(\Sigma)/\bar{\mathcal{E}}$.
- (2) $\mathbf{Ter}^{\mathbf{sp}}$ sends a specification morphism \mathbf{d} from (Σ, \mathcal{E}) to (Λ, \mathcal{H}) to the functor $\mathbf{Ter}^{\mathbf{sp}}(\mathbf{d})$, also occasionally denoted by \mathbf{d}_{\diamond} , from the quotient category $\mathbf{Ter}(\Sigma, \mathcal{E}) = \mathbf{Ter}(\Sigma)/\bar{\mathcal{E}}$ to the quotient category $\mathbf{Ter}(\Lambda, \mathcal{H}) = \mathbf{Ter}(\Lambda)/\bar{\mathcal{H}}$, which assigns to a morphism $[P]_{\bar{\mathcal{E}}}$ from X to Y in $\mathbf{Ter}(\Sigma, \mathcal{E})$ the morphism

$$\mathbf{Ter}^{\mathbf{sp}}(\mathbf{d})([P]_{\bar{\mathcal{E}}}) = [\mathbf{d}_{\diamond}(P)]_{\bar{\mathcal{H}}}: \coprod_{\varphi} X \longrightarrow \coprod_{\varphi} Y$$

in $\mathbf{Ter}(\Lambda, \mathcal{H})$.

Proof. Everything follows, essentially, from the fact that the action of $\mathbf{Ter}^{\mathbf{sp}}(\mathbf{d})$ on $[P]_{\bar{\mathcal{E}}}$ is well defined because $\mathcal{E} \subseteq \mathbf{Ker}(\mathbf{Pr}_{\bar{\mathcal{H}}} \circ \mathbf{d}_{\diamond})$, where $\mathbf{Pr}_{\bar{\mathcal{H}}}$ is the projection functor from $\mathbf{Ter}(\Lambda)$ to the quotient category $\mathbf{Ter}(\Lambda)/\bar{\mathcal{H}}$. \square

After this we prove that the family of functors $\mathbf{Tr} = (\mathbf{Tr}^{\Sigma})_{\Sigma \in \mathbf{Sig}}$, defined in Proposition 18, can be lifted to the family of functors $\mathbf{Tr}^{\mathbf{sp}} = (\mathbf{Tr}^{\mathbf{sp}, (\Sigma, \mathcal{E})})_{(\Sigma, \mathcal{E}) \in \mathbf{Spf}}$.

Proposition 37. *Let (Σ, \mathcal{E}) be a specification. Then from the product category $\mathbf{Alg}(\Sigma, \mathcal{E}) \times \mathbf{Ter}(\Sigma, \mathcal{E})$ to the category \mathbf{Set} there exists a functor $\mathbf{Tr}^{\mathbf{sp}, (\Sigma, \mathcal{E})}$ defined as follows*

- (1) $\mathbf{Tr}^{\mathbf{sp}, (\Sigma, \mathcal{E})}$ sends a pair (\mathbf{A}, X) , formed by a Σ -algebra \mathbf{A} which satisfies \mathcal{E} and an S -sorted set X , to the set $\mathbf{Tr}^{\mathbf{sp}, (\Sigma, \mathcal{E})}(\mathbf{A}, X) = A_X$ of the S -sorted mappings from X to the underlying S -sorted set A of \mathbf{A} ,
- (2) $\mathbf{Tr}^{\mathbf{sp}, (\Sigma, \mathcal{E})}$ sends an arrow $(f, [P]_{\bar{\mathcal{E}}})$ from (\mathbf{A}, X) to (\mathbf{B}, Y) to the mapping $\mathbf{Tr}^{\mathbf{sp}, (\Sigma, \mathcal{E})}(f, [P]_{\bar{\mathcal{E}}}) = f_P$ from A_X to B_Y .

Proof. Everything follows from the fact that the action of $\mathrm{Tr}^{\mathrm{sp},(\Sigma,\mathcal{E})}$ on $(f, [P]_{\bar{\mathcal{E}}})$ is well defined because from $[P]_{\bar{\mathcal{E}}} = [Q]_{\bar{\mathcal{E}}}$ it follows that, for every Σ -algebra \mathbf{C} which satisfies \mathcal{E} , $P^{\mathbf{C}} = Q^{\mathbf{C}}$. \square

Next we state that the family of natural isomorphisms $\theta = (\theta^{\mathbf{d}})_{\mathbf{d} \in \mathrm{Mor}(\mathbf{Sig})}$, defined in Proposition 19, can be lifted to the family of natural isomorphisms $\theta^{\mathrm{sp}} = (\theta^{\mathrm{sp},\mathbf{d}})_{\mathbf{d} \in \mathrm{Mor}(\mathbf{Spf})}$.

Proposition 38. *Let \mathbf{d} be a specification morphism from (Σ, \mathcal{E}) to (Λ, \mathcal{H}) . Then there exists a natural isomorphism $\theta^{\mathrm{sp},\mathbf{d}} = (\theta_{\mathbf{A},X}^{\mathrm{sp},\mathbf{d}})_{(\mathbf{A},X) \in \mathbf{Alg}(\Lambda, \mathcal{H}) \times \mathbf{Ter}(\Sigma, \mathcal{E})}$ as shown in the following diagram*

$$\begin{array}{ccc}
 \mathbf{Alg}(\Lambda, \mathcal{H}) \times \mathbf{Ter}(\Sigma, \mathcal{E}) & \xrightarrow{\mathbf{d}^* \times \mathrm{Id}} & \mathbf{Alg}(\Sigma, \mathcal{E}) \times \mathbf{Ter}(\Sigma, \mathcal{E}) \\
 \mathrm{Id} \times \mathbf{d}_\circ \downarrow & \theta^{\mathrm{sp},\mathbf{d}} \rightrightarrows & \downarrow \mathrm{Tr}^{\mathrm{sp},(\Sigma,\mathcal{E})} \\
 \mathbf{Alg}(\Lambda, \mathcal{H}) \times \mathbf{Ter}(\Lambda, \mathcal{H}) & \xrightarrow{\mathrm{Tr}^{\mathrm{sp},(\Lambda,\mathcal{H})}} & \mathbf{Set}
 \end{array}$$

where, for every $(\mathbf{A}, X) \in \mathbf{Alg}(\Lambda, \mathcal{H}) \times \mathbf{Ter}(\Sigma, \mathcal{E})$, $\theta_{\mathbf{A},X}^{\mathrm{sp},\mathbf{d}}$ is $\theta_{X,A}^\varphi$, i.e., the value at (X, A) of the natural isomorphism of the adjunction $\coprod_{\varphi} \dashv \Delta_\varphi$.

From these two last propositions it follows immediately the following

Corollary 10. *The quadruple $\mathfrak{Spf} = (\mathbf{Spf}, \mathbf{Alg}^{\mathrm{sp}}, \mathbf{Ter}^{\mathrm{sp}}, (\mathrm{Tr}^{\mathrm{sp}}, \theta^{\mathrm{sp}}))$ is an institution on the category \mathbf{Set} , the so-called many-sorted specification institution, or, to abbreviate, the specification institution.*

From the contravariant functor $\mathbf{Alg}^{\mathrm{sp}}$, from \mathbf{Spf} to \mathbf{Cat} , to the contravariant functor $\mathbf{Alg} \circ \mathrm{sig}^{\mathrm{op}}$, between the same categories, there exists a natural transformation, In , which sends a specification (Σ, \mathcal{E}) to the full embedding $\mathrm{In}_{(\Sigma, \mathcal{E})}$ of $\mathbf{Alg}(\Sigma, \mathcal{E})$ into $\mathbf{Alg}(\Sigma)$. Besides, from the pseudo-functor $\mathbf{Ter} \circ \mathrm{sig}$, from \mathbf{Spf} to \mathbf{Cat} , to the pseudo-functor $\mathbf{Ter}^{\mathrm{sp}}$, between the same categories, there exists a (strict) pseudo-natural transformation, Pr , given by the following data

- (1) For each specification (Σ, \mathcal{E}) , the projection functor $\mathrm{Pr}_{\bar{\mathcal{E}}}$ from $\mathbf{Ter}(\Sigma)$ to the quotient category $\mathbf{Ter}(\Sigma)/\bar{\mathcal{E}}$.
- (2) For each specification morphism \mathbf{d} from (Σ, \mathcal{E}) to (Λ, \mathcal{H}) , the identity natural transformation, denoted in this case by $\mathrm{Pr}_{\mathbf{d}}$, from the functor $\mathrm{Pr}_{\bar{\mathcal{H}}} \circ (\mathbf{Ter} \circ \mathrm{sig})(\mathbf{d})$ to the functor $\mathbf{Ter}^{\mathrm{sp}}(\mathbf{d}) \circ \mathrm{Pr}_{\bar{\mathcal{E}}}$, both from $\mathbf{Ter}(\Sigma)$ to $\mathbf{Ter}(\Lambda)/\bar{\mathcal{H}}$.

Therefore we have obtained the following

Corollary 11. *The pair $(\mathrm{sig}, (\mathrm{In}, \mathrm{Pr}))$ is a morphism of institutions from the many-sorted specification institution \mathfrak{Spf} to the many-sorted term institution \mathfrak{Tm} .*

Remark. Since, obviously, the many-sorted term institution \mathfrak{Tm} is embedded in the many-sorted specification institution \mathfrak{Spf} , taking into account the just stated corollary, we can assert that \mathfrak{Tm} is a retract of \mathfrak{Spf} .

4. HALL AND BÉNABOU ALGEBRAS.

The concept of many-sorted clone, that generalizes both that of single-sorted clone axiomatized by P. Hall as a single-sorted partial algebra subject to satisfy some laws (see e.g., [14], pp. 127 and 132, or [46], pp. 136 and 143) and by M. Lazard as a compositor (see [41], p. 327), and that of Boolean clone investigated, among others, by E. Post (see e.g., [52] and [53]), was axiomatically defined by

Goguen and Meseguer (in [26], pp. 318–319) as any many-sorted algebra (of the appropriate signature) that satisfies a system of many-sorted equational laws. The corresponding categories of many-sorted algebras, called categories of Hall algebras, are the algebraic rendering of the categories of finitary many-sorted algebraic theories of Bénabou, i.e., both types of categories, as is well-known, are equivalent.

Our main aim in this section is to define, for each set of sorts, through a system of many-sorted equational laws the, so-called, Bénabou algebras as those many-sorted algebras that satisfy them, and to prove that the corresponding category of Bénabou algebras, for a given set of sorts, is *isomorphic* to the category of finitary many-sorted algebraic theories of Bénabou, for the same set of sorts. Besides, we prove, directly, that the Hall and Bénabou algebras, even having different specification, are models of the essential properties of the clones for the many-sorted operations, i.e., that the respective categories of Hall and Bénabou algebras are *equivalent*.

The homomorphisms between Bénabou algebras, as we will show later on (in the fifth section), are also adequate to define the composition of the morphisms of Fujiwara between signatures, that are a strict generalization of both the standard morphisms and the derivors (defined in the fifth section) between signatures. Informally, we can say that the Bénabou algebras are to the composition of morphisms of Fujiwara between signatures as the Hall algebras are to the composition of derivors between signatures.

Before we define the Hall algebras as the models of a specification, we agree that for a set of sorts U , a word $x \in U^*$ and a standard U -sorted set of variables $V^U = (\{v_n^u \mid n \in \mathbb{N}\})_{u \in U}$, $\downarrow x$ is the U -sorted subset of V^U defined, for every $u \in U$ as $(\downarrow x)_u = \{v_i^u \mid i \in x^{-1}[u]\}$, this will apply, in particular, when $U = S^* \times S$ or $U = S^* \times S^*$.

Definition 24. Let S be a set of sorts and V^{H_S} the $S^* \times S$ -sorted set of variables $(V_{w,s})_{(w,s) \in S^* \times S}$ where, for every $(w, s) \in S^* \times S$, $V_{w,s} = \{v_n^{w,s} \mid n \in \mathbb{N}\}$. A *Hall algebra for S* is a $H_S = (S^* \times S, \Sigma^{H_S}, \mathcal{E}^{H_S})$ -algebra, where Σ^{H_S} is the $S^* \times S$ -sorted signature, i.e., the $(S^* \times S)^* \times (S^* \times S)$ -sorted set, defined as follows:

HS₁. For every $w \in S^*$ and $i \in |w|$,

$$\pi_i^w: \lambda \longrightarrow (w, w_i),$$

where $|w|$ is the *length* of the word w and λ the *empty word* in $(S^* \times S)^*$.

HS₂. For every $u, w \in S^*$ and $s \in S$,

$$\xi_{u,w,s}: ((w, s), (u, w_0), \dots, (u, w_{|w|-1})) \longrightarrow (u, s);$$

while \mathcal{E}^{H_S} is the ms-subset of $\text{Eq}(\Sigma^{H_S}) = (\text{T}_{\Sigma^{H_S}}(\downarrow \bar{w})_{(u,s)}^2)_{(\bar{w}, (u,s)) \in (S^* \times S)^* \times (S^* \times S)}$ defined as follows:

H₁. *Projection*. For every $u, w \in S^*$ and $i \in |w|$, the equation

$$\xi_{u,w,w_i}(\pi_i^w, v_0^{u,w_0}, \dots, v_{|w|-1}^{u,w_{|w|-1}}) = v_i^{u,w_i}$$

of type $((u, w_0), \dots, (u, w_{|w|-1}), (u, w_i))$.

H₂. *Identity*. For every $u \in S^*$ and $j \in |u|$, the equation

$$\xi_{u,u,u_j}(v_j^{u,u_j}, \pi_0^u, \dots, \pi_{|u|-1}^u) = v_j^{u,u_j}$$

of type $((u, u_j), (u, u_j))$.

H₃. *Associativity.* For every $u, v, w \in S^*$ and $s \in S$, the equation

$$\begin{aligned} \xi_{u,v,s}(\xi_{v,w,s}(v_0^{w,s}, v_1^{v,w_0}, \dots, v_{|w|}^{v,w_{|w|-1}}), v_{|w|+1}^{u,v_0}, \dots, v_{|w|+|v|-1}^{u,v_{|v|-1}}) = \\ \xi_{u,w,s}(v_0^{w,s}, \xi_{u,v,w_0}(v_1^{v,w_0}, v_{|w|+1}^{u,v_0}, \dots, v_{|w|+|v|-1}^{u,v_{|v|-1}}), \dots, \\ \xi_{u,v,w_{|w|-1}}(v_{|w|}^{v,w_{|w|-1}}, v_{|w|+1}^{u,v_0}, \dots, v_{|w|+|v|-1}^{u,v_{|v|-1}})) \end{aligned}$$

of type $((w, s), (v, w_0), \dots, (v, w_{|w|-1}), (u, v_0), \dots, (u, v_{|v|-1}), (u, s))$.

Remark. From H₃, for $w = \lambda$, we get the invariance of constant functions axiom in [26]: For every $u, v \in S^*$ and $s \in S$, we have the equation

$$\xi_{u,v,s}(\xi_{v,\lambda,s}(v_0^{\lambda,s}), v_1^{u,v_0}, \dots, v_{|v|}^{u,v_{|v|-1}}) = \xi_{u,\lambda,s}(v_0^{\lambda,s})$$

of type $((\lambda, s), (u, v_0), \dots, (u, v_{|v|-1}), (u, s))$.

We call the formal constants π_i^w *projections*, and the formal operations $\xi_{u,w,s}$ *substitution operators*. Furthermore, we denote by $\mathbf{Alg}(\mathbf{H}_S)$ the category of Hall algebras for S and homomorphisms between Hall algebras. Since $\mathbf{Alg}(\mathbf{H}_S)$ is a variety, the forgetful functor $\mathbf{G}_{\mathbf{H}_S}$ from $\mathbf{Alg}(\mathbf{H}_S)$ to $\mathbf{Set}^{S^* \times S}$ has a left adjoint $\mathbf{T}_{\mathbf{H}_S}$

$$\mathbf{Alg}(\mathbf{H}_S) \begin{array}{c} \xrightarrow{\mathbf{G}_{\mathbf{H}_S}} \\ \top \\ \xleftarrow{\mathbf{T}_{\mathbf{H}_S}} \end{array} \mathbf{Set}^{S^* \times S}$$

which assigns to an $S^* \times S$ -sorted set Σ the corresponding free Hall algebra $\mathbf{T}_{\mathbf{H}_S}(\Sigma)$.

For every S -sorted set A , $\mathbf{HOp}_S(A) = (\mathbf{Hom}(A_w, A_s))_{(w,s) \in S^* \times S}$, the $S^* \times S$ -sorted set of operation for A , is naturally endowed with a structure of Hall algebra, as stated in the following proposition, if we realize the projections as the true projections and the substitution operators as the generalized composition of mappings.

Proposition 39. *Let A be an S -sorted set and $\mathbf{HOp}_S(A)$ the $\Sigma^{\mathbf{H}_S}$ -algebra with underlying ms-set $\mathbf{HOp}_S(A)$ and algebraic structure defined as follows*

- (1) For every $w \in S^*$ and $i \in |w|$, $(\pi_i^w)^{\mathbf{HOp}_S(A)} = \text{pr}_{w,i}^A : A_w \longrightarrow A_{w_i}$.
- (2) For every $u, w \in S^*$ and $s \in S$, $\xi_{u,w,s}^{\mathbf{HOp}_S(A)}$ is defined, for every $f \in A_s^{A_w}$ and $g \in A_w^{A_u}$, as $\xi_{u,w,s}^{\mathbf{HOp}_S(A)}(f, g_0, \dots, g_{|w|-1}) = f \circ \langle g_i \rangle_{i \in |w|}$, where $\langle g_i \rangle_{i \in |w|}$ is the unique mapping from A_u to A_w such that, for every $i \in |w|$, $\text{pr}_{w,i}^A \circ \langle g_i \rangle_{i \in |w|} = g_i$.

Then $\mathbf{HOp}_S(A)$ is a Hall algebra, the Hall algebra for (S, A) .

Remark. The closed sets of the Hall algebra $\mathbf{HOp}_S(A)$ for (S, A) are precisely the clones of (many-sorted) operations on the S -sorted set A .

For every S -sorted signature Σ , $\mathbf{HTer}_S(\Sigma) = (\mathbf{T}_\Sigma(\downarrow w)_s)_{(w,s) \in S^* \times S}$ is also endowed with a structure of Hall algebra that formalizes the concept of substitution as stated in the following

Proposition 40. *Let Σ be an S -sorted signature and $\mathbf{HTer}_S(\Sigma)$ the $\Sigma^{\mathbf{H}_S}$ -algebra with underlying ms-set $\mathbf{HTer}_S(\Sigma)$ and algebraic structure defined as follows*

- (1) For every $w \in S^*$ and $i \in |w|$, $(\pi_i^w)^{\mathbf{HTer}_S(\Sigma)}$ is the image under $\eta_{\downarrow w, w_i}$ of the variable $v_i^{w_i}$, where $\eta_{\downarrow w} = (\eta_{\downarrow w, s})_{s \in S}$ is the canonical embedding of $\downarrow w$ into $\mathbf{T}_\Sigma(\downarrow w)$.
- (2) For every $u, w \in S^*$ and $s \in S$, $\xi_{u,w,s}^{\mathbf{HTer}_S(\Sigma)}$ is the mapping

$$\xi_{u,w,s}^{\mathbf{HTer}_S(\Sigma)} \left\{ \begin{array}{l} \mathbf{T}_\Sigma(\downarrow w)_s \times \mathbf{T}_\Sigma(\downarrow u)_{w_0} \times \dots \times \mathbf{T}_\Sigma(\downarrow u)_{w_{|w|-1}} \longrightarrow \mathbf{T}_\Sigma(\downarrow u)_s \\ (P, (Q_i \mid i \in |w|)) \longmapsto \mathcal{Q}_s^\sharp(P) \end{array} \right.$$

where, for \mathcal{Q} the S -sorted mapping from $\downarrow w$ to $\mathbf{T}_\Sigma(\downarrow u)$ canonically associated to the family $(Q_i \mid i \in |w|)$, \mathcal{Q}^\sharp is the unique homomorphism from $\mathbf{T}_\Sigma(\downarrow w)$ into $\mathbf{T}_\Sigma(\downarrow u)$ such that $\mathcal{Q}^\sharp \circ \eta_{\downarrow w} = \mathcal{Q}$.

Then $\mathbf{HTer}_S(\Sigma)$ is a Hall algebra, the Hall algebra for (S, Σ) .

Our next goal is to prove that, for every $S^* \times S$ -sorted set Σ , $\mathbf{T}_{\mathbf{H}_S}(\Sigma)$, the free Hall algebra on Σ , is isomorphic to $\mathbf{HTer}_S(\Sigma)$. We remark that the existence of this isomorphism is interesting because it enables us, on the one hand, to get a more tractable description of the terms in $\mathbf{T}_{\mathbf{H}_S}(\Sigma)$, and, on the other hand, to give, in the fifth section, an alternative, but equivalent, definition of the concept of derivor (defined by Goguen, Thatcher and Wagner in [27], p. 86) between signatures.

To attain the goal just stated we define, for a Hall algebra \mathbf{A} , an S -sorted signature Σ , an $S^* \times S$ -mapping $f: \Sigma \longrightarrow A$, and a word $u \in S^*$, the concept of derived Σ -algebra of \mathbf{A} for (f, u) , since it will be used afterwards in the proof of the isomorphism between $\mathbf{T}_{\mathbf{H}_S}(\Sigma)$ and $\mathbf{HTer}_S(\Sigma)$.

Definition 25. Let \mathbf{A} be a Hall algebra and Σ an S -sorted signature. Then, for every $f: \Sigma \longrightarrow A$ and $u \in S^*$, $\mathbf{A}^{f,u}$, the derived Σ -algebra of \mathbf{A} for (f, u) , is the Σ -algebra with underlying S -sorted set $A^{f,u} = (A_{u,s})_{s \in S}$ and algebraic structure $F^{f,u}$, defined, for every $(w, s) \in S^* \times S$, as

$$F_{w,s}^{f,u} \begin{cases} \Sigma_{w,s} \longrightarrow \mathbf{HOp}_w(A^{f,u})_s \\ \sigma \longmapsto \begin{cases} \prod_{i \in |w|} A_{u,w_i} \longrightarrow A_{u,s} \\ (a_0, \dots, a_{|w|-1}) \longmapsto \xi_{u,w,s}^{\mathbf{A}}(f_{(w,s)}(\sigma), a_0, \dots, a_{|w|-1}) \end{cases} \end{cases}$$

where $\mathbf{HOp}_w(A^{f,u})_s = A_{u,s}^{\prod_{i \in |w|} A_{u,w_i}}$. Furthermore, we denote by p^u the S -sorted mapping from $\downarrow u$ to $A^{f,u}$ defined, for every $s \in S$ and $i \in |u|$, as $p_s^u(v_i^s) = (\pi_i^u)^{\mathbf{A}}$, and by $(p^u)^\sharp$ the unique homomorphism from $\mathbf{T}_\Sigma(\downarrow u)$ to $\mathbf{A}^{f,u}$ such that $(p^u)^\sharp \circ \eta_{\downarrow u} = p^u$.

Remark. For a Σ -algebra $\mathbf{B} = (B, G)$, we have that $G: \Sigma \longrightarrow \mathbf{HOp}_S(B)$ and $\mathbf{B} \cong \mathbf{HOp}_S(B)^{G,\lambda}$, where λ is the empty word on S . Besides, for every $u \in S^*$, we have that $\mathbf{Op}_{\downarrow u}(\mathbf{B}) \cong \mathbf{HOp}_S(B)^{G,u}$.

Lemma 6. Let Σ be an S -sorted signature, \mathbf{A} a Hall algebra, $f: \Sigma \longrightarrow A$ and $u \in S^*$. Then, for every $(w, s) \in S^* \times S$, $P \in \mathbf{T}_\Sigma(\downarrow w)_s$ and $a \in \prod_{i \in |w|} A_{u,w_i}$, we have that

$$P^{\mathbf{A}^{f,u}}(a_0, \dots, a_{|w|-1}) = \xi_{u,w,s}^{\mathbf{A}}((p^w)_s^\sharp(P), a_0, \dots, a_{|w|-1}).$$

Proof. By algebraic induction on the complexity of P . If P is a variable v_i^s , with $i \in |w|$, then

$$\begin{aligned} v_i^{s,\mathbf{A}^{f,u}}(a_0, \dots, a_{|w|-1}) &= a_{w_i}^\sharp(v_i^s) \\ &= a_i \\ &= \xi_{u,w,s}^{\mathbf{A}}((\pi_i^w)^{\mathbf{A}}, a_0, \dots, a_{|w|-1}) \quad (\mathbf{H}_1) \\ &= \xi_{u,w,s}^{\mathbf{A}}((p^w)_s^\sharp(v_i^s), a_0, \dots, a_{|w|-1}). \end{aligned}$$

Let us assume that $P = \sigma(Q_0, \dots, Q_{|x|-1})$, with $\sigma: x \longrightarrow s$ and that, for every $j \in |x|$, $Q_j \in \mathbf{T}_\Sigma(\downarrow w)_{x_j}$ fulfills the induction hypothesis. Then we have that

$$\begin{aligned}
& (\sigma(Q_0, \dots, Q_{|x|-1}))^{\mathbf{A}^{f,u}}(a_0, \dots, a_{|w|-1}) \\
&= \sigma^{\mathbf{A}^{f,u}}(Q_0^{\mathbf{A}^{f,u}}(a_0, \dots, a_{|w|-1}), \dots, Q_{|x|-1}^{\mathbf{A}^{f,u}}(a_0, \dots, a_{|w|-1})) \\
&= \xi_{u,x,s}^{\mathbf{A}}(f(\sigma), Q_0^{\mathbf{A}^{f,u}}(a_0, \dots, a_{|w|-1}), \dots, Q_{|x|-1}^{\mathbf{A}^{f,u}}(a_0, \dots, a_{|w|-1})) \\
&= \xi_{u,x,s}^{\mathbf{A}}(f(\sigma), \xi_{u,w,x_0}^{\mathbf{A}}((p^w)_{x_0}^\sharp(Q_0), a_0, \dots, a_{|w|-1}), \dots, \\
&\quad \xi_{u,w,x_{|x|-1}}^{\mathbf{A}}((p^w)_{x_{|x|-1}}^\sharp(Q_{|x|-1}), a_0, \dots, a_{|w|-1})) \quad (\text{Ind. hyp.}) \\
&= \xi_{u,w,s}^{\mathbf{A}}(\xi_{w,x,s}^{\mathbf{A}}(f(\sigma), (p^w)_{x_0}^\sharp(Q_0), \dots, (p^w)_{x_{|x|-1}}^\sharp(Q_{|x|-1})), a_0, \dots, a_{|w|-1}) \quad (\text{H3}) \\
&= \xi_{u,w,s}^{\mathbf{A}}(\sigma^{\mathbf{A}^w}((p^w)_{x_0}^\sharp(Q_0), \dots, (p^w)_{x_{|x|-1}}^\sharp(Q_{|x|-1})), a_0, \dots, a_{|w|-1}) \\
&= \xi_{u,w,s}^{\mathbf{A}}((p^w)_s^\sharp(\sigma, Q_0, \dots, Q_{|x|-1}), a_0, \dots, a_{|w|-1}) \\
&= \xi_{u,w,s}^{\mathbf{A}}((p^w)_s^\sharp(P), a_0, \dots, a_{|w|-1}). \quad \square
\end{aligned}$$

Next we prove, as announced above, that, for every $S^* \times S$ -sorted set Σ , the Hall algebra for (S, Σ) is isomorphic to the free Hall algebra on Σ .

Proposition 41. *Let Σ be an S -sorted signature, i.e., an $S^* \times S$ -sorted set. Then the Hall algebra $\mathbf{HTer}_S(\Sigma)$ is isomorphic to $\mathbf{T}_{\text{HS}}(\Sigma)$.*

Proof. It is enough to prove that $\mathbf{HTer}_S(\Sigma)$ has the universal property of the free Hall algebra on Σ . Therefore we have to specify an $S^* \times S$ -sorted mapping h from Σ to $\mathbf{HTer}_S(\Sigma)$ such that, for every Hall algebra \mathbf{A} and $S^* \times S$ -sorted mapping f from Σ to \mathbf{A} , there is a unique homomorphism \widehat{f} from $\mathbf{HTer}_S(\Sigma)$ to \mathbf{A} such that $\widehat{f} \circ h = f$. Let h be the $S^* \times S$ -sorted mapping defined, for every $(w, s) \in S^* \times S$, as

$$h_{w,s} \begin{cases} \Sigma_{w,s} & \longrightarrow \mathbf{T}_\Sigma(\downarrow w)_s \\ \sigma & \longmapsto \sigma(v_0^s, \dots, v_{|w|-1}^s) \end{cases}$$

Let \mathbf{A} be a Hall algebra, $f: \Sigma \longrightarrow \mathbf{A}$ an $S^* \times S$ -sorted mapping and \widehat{f} the $S^* \times S$ -sorted mapping from $\mathbf{HTer}_S(\Sigma)$ to \mathbf{A} defined, for every $(w, s) \in S^* \times S$, as $\widehat{f}_{w,s} = (p^w)_s^\sharp$, where, we recall, $(p^w)^\sharp$ is the unique homomorphism from $\mathbf{T}_\Sigma(\downarrow w)$ to $\mathbf{A}^{f,w}$ such that the following diagram commutes

$$\begin{array}{ccc} \downarrow w & \xrightarrow{\eta_{\downarrow w}} & \mathbf{T}_\Sigma(\downarrow w) \\ & \searrow p^w & \downarrow (p^w)^\sharp \\ & & (\mathbf{A}_{w,s})_{s \in S} \end{array}$$

Then \widehat{f} is a homomorphism of Hall algebras, because, on the one hand, for $w \in S^*$ and $i \in |w|$ we have that

$$\begin{aligned} \widehat{f}_{w,w_i}((\pi_i^w)^{\mathbf{HTer}_S(\Sigma)}) &= \widehat{f}_{w,w_i}(v_i^s) \\ &= p_{w_i}^w(v_i^s) \\ &= (\pi_i^w)^{\mathbf{A}}, \end{aligned}$$

and, on the other hand, for $P \in \mathbf{T}_\Sigma(\downarrow w)_s$ and $(Q_i \mid i \in |w|) \in \mathbf{T}_\Sigma(\downarrow w)_w$ we have that

$$\begin{aligned}
& \widehat{f}_{u,s}(\xi_{u,w,s}^{\mathbf{HTer}_S(\Sigma)}(P, Q_0, \dots, Q_{|w|-1})) \\
&= (p^u)_s^\#(\mathcal{Q}_s^\#(P)) \\
&= ((p^u)^\# \circ \mathcal{Q})_s^\#(P) \quad (\text{because } (p^u)^\# \circ \mathcal{Q}^\# = ((p^u)^\# \circ \mathcal{Q})^\#) \\
&= P^{\mathbf{A}^{f,u}}((p^u)_{w_0}^\#(Q_0), \dots, (p^u)_{w_{|w|-1}}^\#(Q_{|w|-1})) \\
&= \xi_{u,w,s}^{\mathbf{A}}((p^w)_s^\#(P), (p^u)_{w_0}^\#(Q_0), \dots, (p^u)_{w_{|w|-1}}^\#(Q_{|w|-1})) \quad (\text{by Lemma 6}) \\
&= \xi_{u,w,s}^{\mathbf{A}}(\widehat{f}_{w,s}(P), \widehat{f}_{u,w_0}(Q_0), \dots, \widehat{f}_{u,w_{|w|-1}}(Q_{|w|-1})).
\end{aligned}$$

Therefore the $S^* \times S$ -sorted mapping \widehat{f} is a homomorphism. Furthermore, $\widehat{f} \circ h = f$, because, for every $w \in S^*$, $s \in S$, and $\sigma \in \Sigma_{w,s}$, we have that

$$\begin{aligned}
\widehat{f}_{w,s}(h_{w,s}(\sigma)) &= (p^w)_s^\#(\sigma(v_0^s, \dots, v_{|w|-1}^s)) \\
&= \sigma^{\mathbf{A}^w}(p_{w_0}^w(v_0^s), \dots, p_{w_{|w|-1}}^w(v_{|w|-1}^s)) \\
&= \xi_{w,w,s}^{\mathbf{A}}(f_{(w,s)}(\sigma), (\pi_0^w)^{\mathbf{A}}, \dots, (\pi_{|w|-1}^w)^{\mathbf{A}}) \\
&= f_{w,s}(\sigma) \quad (\mathbf{H}_2).
\end{aligned}$$

It is obvious that \widehat{f} is the unique homomorphism such that $\widehat{f} \circ h = f$. Henceforth $\mathbf{HTer}_S(\Sigma)$ is isomorphic to $\mathbf{T}_{H_S}(\Sigma)$. \square

This isomorphism together with the adjunction $\mathbf{T}_{H_S} \dashv \mathbf{G}_{H_S}$ has as a consequence that, for every S -sorted set A and S -sorted signature Σ , the sets $\text{Hom}(\Sigma, \mathbf{HOp}_S(A))$ and $\text{Hom}(\mathbf{HTer}_S(\Sigma), \mathbf{HOp}_S(A))$ are naturally isomorphic. Actually, the isomorphism sends, for an S -sorted set A , a structure of Σ -algebra F on A , i.e., an ms-mapping F from Σ to $\mathbf{HOp}_S(A)$, to the homomorphism of Hall algebras $\mathbf{HTr}_S^{(A,F)} = (\text{Tr}_s^{\downarrow w, (A,F)})_{(w,s) \in S^* \times S}$, where, for every $w \in S^*$, $\text{Tr}_s^{\downarrow w, (A,F)} = (\text{Tr}_s^{\downarrow w, (A,F)})_{s \in S}$ is the unique homomorphism from $\mathbf{T}_\Sigma(\downarrow w)$ to $\mathbf{Op}_{\downarrow w}(A, F) \cong (A, F)^{A^w}$ such that the following diagram commutes

$$\begin{array}{ccc}
\downarrow w & \xrightarrow{\eta_{\downarrow w}} & \mathbf{T}_\Sigma(\downarrow w) \\
& \searrow \text{p}_{\downarrow w}^A & \downarrow \text{Tr}^{\downarrow w, (A,F)} \\
& & \mathbf{Op}_{\downarrow w}(A, F)
\end{array}$$

where $\text{p}_{\downarrow w}^A$ is the S -sorted mapping defined, for every $s \in S$ and $v_i^s \in (\downarrow w)_s$, as $\text{p}_{\downarrow w,s}^A(v_i^s) = \text{pr}_{w,i}^A$; while the inverse isomorphism sends an homomorphism h from $\mathbf{HTer}_S(\Sigma)$ to $\mathbf{HOp}_S(A)$ to the algebraic structure $\mathbf{G}_{H_S}(h) \circ \eta_\Sigma$ on A .

For a set of sorts S , the fundamental objects in the approach to the many-sorted completeness theorem in [26], i.e., the Hall algebras for S , have an alternative, but equivalent, description in terms of, what we call, Bénabou algebras for S , that, as we will show below are more strongly linked to the finitary many-theories algebraic theories than are the Hall algebras. Besides, the Bénabou algebras will be shown to be more adequate in order to work with morphisms between signatures more general than the standard ones. Actually there exists an equivalence between the category $\mathbf{Alg}(\mathbf{H}_S)$, of Hall algebras for S , and the category $\mathbf{Alg}(\mathbf{B}_S)$, of Bénabou algebras for S , i.e., the category defined as follows.

Definition 26. Let S be a set of sorts and V^{B_S} the $(S^*)^2$ -sorted set of variables $(V_{u,w})_{(u,w) \in (S^*)^2}$ where, for every $(u,w) \in (S^*)^2$, $V_{u,w} = \{v_n^{u,w} \mid n \in \mathbb{N}\}$.

A *Bénabou algebra* for S is a $B_S = ((S^*)^2, \Sigma^{B_S}, \mathcal{E}^{B_S})$ -algebra, where Σ^{B_S} is the $(S^*)^2$ -sorted signature defined as follows:

BS₁. For the empty word $\lambda \in S^*$, every $w \in S^*$ and $i \in |w|$, where $|w|$ is the domain of the word w , the formal operation of *projection*:

$$\pi_i^w : \lambda \longrightarrow (w, (w_i)).$$

BS₂. For every $u, w \in S^*$, the formal operation of *finite tupling*:

$$\langle \rangle_{u,w} : ((u, (w_0)), \dots, (u, (w_{|w|-1}))) \longrightarrow (u, w).$$

BS₃. For every $u, x, w \in S^*$, the formal operation of *substitution*:

$$\circ_{u,x,w} : ((u, x), (x, w)) \longrightarrow (u, w);$$

while \mathcal{E}^{B_S} is the ms-subset of $\text{Eq}(\Sigma^{B_S}) = (\mathbf{T}_{\Sigma^{B_S}}(\downarrow \bar{w})_{(u,x)}^2)_{(\bar{w}, (u,x)) \in ((S^*)^2)^* \times (S^*)^2}$ defined as follows:

B₁. For every $u, w \in S^*$ and $i \in |w|$, the equation:

$$\pi_i^w \circ_{u,w,(w_i)} \langle v_0^{u,(w_0)}, \dots, v_{|w|-1}^{u,(w_{|w|-1})} \rangle_{u,w} = v_i^{u,(w_i)},$$

of type $((u, (w_0)), \dots, (u, (w_{|w|-1}))), (u, (w_i))$.

B₂. For every $u, w \in S^*$, the equation:

$$v_0^{u,w} \circ_{u,u,w} \langle \pi_0^u, \dots, \pi_{|u|-1}^u \rangle_{u,u} = v_0^{u,w},$$

of type $((u, w), (u, w))$.

B₃. For every $u, w \in S^*$, the equation:

$$\langle \pi_0^w \circ_{u,w,w_0} v_0^{u,w}, \dots, \pi_{|w|-1}^w \circ_{u,w,w_{|w|-1}} v_0^{u,w} \rangle_{u,w} = v_0^{u,w},$$

of type $((u, w), (u, w))$.

B₄. For every $w \in S^*$, the equation:

$$\langle \pi_0^w \rangle_{w,(w_0)} = \pi_0^w,$$

of type $((w, (w_0)), (w, (w_0)))$.

B₅. For every $u, x, w, y \in S^*$, the equation:

$$v_0^{w,y} \circ_{u,w,y} (v_1^{x,w} \circ_{u,x,w} v_2^{u,x}) = (v_0^{w,y} \circ_{x,w,y} v_1^{x,w}) \circ_{u,x,y} v_2^{u,x},$$

of type $((w, y), (x, w), (u, x)), (u, y)$,

where $v_n^{u,w}$ is the n -th variable of type (u, w) , $Q \circ_{u,x,w} P$ is $\circ_{u,x,w}(P, Q)$, and $\langle P_0, \dots, P_{|w|-1} \rangle_{u,w}$ is $\langle \rangle_{u,w}(P_0, \dots, P_{|w|-1})$.

Since $\mathbf{Alg}(B_S)$ is a variety, the forgetful functor G_{B_S} from $\mathbf{Alg}(B_S)$ to $\mathbf{Set}^{S^* \times S^*}$ has a left adjoint \mathbf{T}_{B_S}

$$\mathbf{Alg}(B_S) \begin{array}{c} \xrightarrow{G_{B_S}} \\ \mathbf{T}_{B_S} \\ \xleftarrow{\quad} \end{array} \mathbf{Set}^{S^* \times S^*}$$

which assigns to an $S^* \times S^*$ -sorted set the corresponding free Bénabou algebra.

For every S -sorted set A , $\mathbf{BOP}_S(A) = (\text{Hom}(A_w, A_u))_{(w,u) \in S^* \times S^*}$ is endowed with a structure of Bénabou algebra as stated in the following

Proposition 42. *Let A be an S -sorted set and $\mathbf{BOP}_S(A)$ the Σ^{B_S} -algebra with underlying many-sorted set $\mathbf{BOP}_S(A)$ and algebraic structure defined as follows*

- (1) *For every $w \in S^*$ and $i \in |w|$, $(\pi_i^w)^{\mathbf{BOP}_S(A)} = \text{pr}_{w,i}^A : A_w \longrightarrow A_{(w_i)}$.*

- (2) For every $u, w \in S^*$, $\langle \rangle_{u,w}^{\mathbf{BOP}_S(A)}$ is defined, for every $(f_0, \dots, f_{|w|-1})$ in $\prod_{i \in |w|} \text{Hom}(A_w, A_{(w_i)})$, as $\langle \rangle_{u,w}^{\mathbf{BOP}_S(A)}(f_0, \dots, f_{|w|-1}) = \langle f_i \rangle_{i \in |w|}$, where $\langle f_i \rangle_{i \in |w|}$ is the unique mapping from A_u to A_w such that, for every $i \in |w|$, $\text{pr}_{w,i}^A \circ \langle f_i \rangle_{i \in |w|} = f_i$.
- (3) For every $u, x, w \in S^*$, $\circ_{u,x,w}^{\mathbf{BOP}_S(A)}$ is defined as the composition of mappings. Then $\mathbf{BOP}_S(A)$ is a Bénabou algebra, the Bénabou algebra for (S, A) .

For every S -sorted signature Σ , $\mathbf{BTer}_S(\Sigma) = (\mathbf{T}_\Sigma(\downarrow w)_u)_{(w,u) \in S^* \times S^*}$, that is naturally isomorphic to $(\text{Hom}(\downarrow u, \mathbf{T}_\Sigma(\downarrow w)))_{(w,u) \in S^* \times S^*}$, is endowed with a structure of Bénabou algebra as stated in the following

Proposition 43. *Let Σ be an S -sorted signature and $\mathbf{BTer}_S(\Sigma)$ the $\Sigma^{\mathbf{B}S}$ -algebra with underlying many-sorted set $\mathbf{BTer}_S(\Sigma)$ and algebraic structure that obtained, by transport of structure, from the algebraic structure defined on the $S^* \times S^*$ -sorted set $(\text{Hom}(\downarrow u, \mathbf{T}_\Sigma(\downarrow w)))_{(w,u) \in S^* \times S^*}$ as follows*

- (1) For every $w \in S^*$ and $i \in |w|$, $(\pi_i^w)^{\mathbf{BTer}_S(\Sigma)}$ is the composition of the canonical embedding from $\downarrow(w_i)$ to $\downarrow w$ and the canonical embedding from $\downarrow w$ to $\mathbf{T}_\Sigma(\downarrow w)$.
- (2) For every $u, w \in S^*$, $\langle \rangle_{u,w}^{\mathbf{BTer}_S(\Sigma)}$ is the canonical isomorphism from the cartesian product $\prod_{i \in |w|} \text{Hom}(\downarrow(w_i), \mathbf{T}_\Sigma(\downarrow u))$ to $\text{Hom}(\downarrow w, \mathbf{T}_\Sigma(\downarrow u))$.
- (3) For every $u, x, w \in S^*$, $\circ_{u,x,w}^{\mathbf{BTer}_S(A)}$ is defined as the mapping which sends a pair $\mathcal{P} \in \text{Hom}(\downarrow x, \mathbf{T}_\Sigma(\downarrow u))$ and $\mathcal{Q} \in \text{Hom}(\downarrow w, \mathbf{T}_\Sigma(\downarrow x))$ to $\mathcal{P}^\sharp \circ \mathcal{Q}$.

Then $\mathbf{BTer}_S(\Sigma)$ is a Bénabou algebra, the Bénabou algebra for (S, Σ) .

Next, after defining the category $\mathbf{BTh}_f(S)$, of finitary many-sorted algebraic theories of Bénabou (defined for the first time in [2]), that generalize the finitary single-sorted algebraic theories of Lawvere, we prove that there exists an isomorphism between the category $\mathbf{BTh}_f(S)$ and the category $\mathbf{Alg}(\mathbf{B}_S)$.

Definition 27. We denote by $\mathbf{BTh}_f(S)$ the category with objects pairs $\mathcal{B} = (\mathbf{B}, p)$, where \mathbf{B} is a category that has as objects the words on S and p a family $(p^w)_{w \in S^*}$ such that, for every word $w \in S^*$, p^w is a family $(p_i^w : w \longrightarrow (w_i))_{i \in |w|}$ of morphisms in \mathbf{B} , the *projections* for w , where (w_i) is the word of length 1 on S whose only letter is w_i , such that (w, p^w) is a product in \mathbf{B} of the family of words $((w_i))_{i \in |w|}$, and as morphisms from \mathcal{B} to \mathcal{B}' functors F from \mathbf{B} to \mathbf{B}' such that the object mapping of F is the identity and the morphism mapping of F preserves the projections, i.e., for every $w \in S^*$ and $i \in |w|$, $F(p_i^{w, \mathcal{B}}) = p_i^{w, \mathcal{B}'}$.

Proposition 44. *There exists an isomorphism from the category $\mathbf{Alg}(\mathbf{B}_S)$ to the category $\mathbf{BTh}_f(S)$.*

Proof. The isomorphism from $\mathbf{Alg}(\mathbf{B}_S)$ to $\mathbf{BTh}_f(S)$ is the functor $B_{a,t}$ which to a Bénabou algebra \mathbf{B} assigns the Bénabou theory $B_{a,t}(\mathbf{B})$ which has as underlying category that given by the following data

- (1) The set of objects is S^* and, for $u, w \in S^*$, $\text{Hom}(u, w) = B_{u,w}$,
- (2) For every $w \in S^*$, $\text{id}_w = \langle (\pi_i^w)^{\mathbf{B}} \mid i \in |w| \rangle_{w,w}$,
- (3) If $P : u \longrightarrow x$, $Q : x \longrightarrow w$, then the composition of P and Q is $\circ_{u,x,w}^{\mathbf{B}}(P, Q)$,

and as underlying family of projections that given, for every $w \in S^*$, as $\pi^w = ((\pi_i^w)^{\mathbf{B}})_{i \in |w|}$; and which to a morphism of Bénabou algebras $f : \mathbf{B} \longrightarrow \mathbf{B}'$ assigns the morphism of Bénabou theories $B_{a,t}(f)$ that to $P : w \longrightarrow u$ associates $f_{w,u}(P) : w \longrightarrow u$.

The inverse of $B_{a,t}$ is the functor $B_{t,a}$ which to a Bénabou theory $\mathcal{B} = (\mathbf{B}, p)$ assigns the Bénabou algebra $B_{t,a}(\mathcal{B})$ that has

- (1) As underlying $(S^*)^2$ -sorted set the family $(\text{Hom}_{\mathbf{B}}(w, u))_{(w,u) \in (S^*)^2}$, and
- (2) As structure of Bénabou algebra on $(\text{Hom}_{\mathbf{B}}(w, u))_{(w,u) \in (S^*)^2}$ that obtained by interpreting, for every $w \in S^*$ and $i \in |w|$, π_i^w as p_i^w , for every $u, w \in S^*$, $\langle \rangle_{u,w}$ as the canonical mapping from $\prod_{i \in |w|} \text{Hom}_{\mathbf{B}}(u, (w_i))$ to $\text{Hom}_{\mathbf{B}}(u, w)$ obtained by the universal property of the product for w , and, for every $u, x, w \in S^*$, $\circ_{u,x,w}$ as the composition in \mathbf{B} ;

and which to a morphism of Bénabou theories $F: \mathbf{B} \longrightarrow \mathbf{B}'$ assigns the morphism of Bénabou algebras $B_{t,a}(F)$, that for every $u, w \in S^*$, is the bi-restriction of F to the corresponding hom-sets $\text{Hom}(u, w)$ and $\text{Hom}(u, w)$. \square

Remark. The isomorphism between $\mathbf{BTh}_f(S)$ and $\mathbf{Alg}(\mathbf{B}_S)$ can be interpreted as meaning, and this can be algebraically reassuring, that the category of finitary many-sorted algebraic theories of Bénabou, a purely formal entity, has the form of a category of models for a finitary many-sorted equational presentation, a semantical, or substantial, entity, therefore confirming, once more, that apparently *form is substance*. Moreover, the isomorphism shows that the Bénabou algebras are more closely related to the finitary many-sorted algebraic theories of Bénabou than are the Hall algebras.

Next we prove, directly, that the categories $\mathbf{Alg}(\mathbf{H}_S)$ and $\mathbf{Alg}(\mathbf{B}_S)$ of Hall and Bénabou algebras, respectively, are equivalent. Later on, once we have defined the morphisms and transformations of Fujiwara, we will get such an equivalence as a consequence of the existence of both an equivalence between the specifications of Hall and Bénabou and a pseudo-functor from the 2-category of specifications to the 2-category \mathbf{Cat} .

Proposition 45. *For every set of sorts S , the categories $\mathbf{Alg}(\mathbf{H}_S)$ and $\mathbf{Alg}(\mathbf{B}_S)$ are equivalent.*

Proof. The equivalence from $\mathbf{Alg}(\mathbf{H}_S)$ to $\mathbf{Alg}(\mathbf{B}_S)$ is the functor $F_{h,b}$ which to a Hall algebra \mathbf{A} assigns the Bénabou algebra $F_{h,b}(\mathbf{A})$ that has

- (1) As underlying $(S^*)^2$ -sorted set $((A_w)_u)_{(w,u) \in (S^*)^2}$ where $A_w = (A_{w,s})_{s \in S}$ and $(A_w)_u = \prod_{i \in |u|} A_{w,u_i}$, and
- (2) As structure of Bénabou algebra on $((A_w)_u)_{(w,u) \in (S^*)^2}$ that defined as

$$\begin{aligned} (\pi_i^w)^{F_{h,b}(\mathbf{A})} &= ((\pi_i^w)^{\mathbf{A}}), \\ \langle (a_0), \dots, (a_{|w|-1}) \rangle_{u,w}^{F_{h,b}(\mathbf{A})} &= (\xi_{u,w,w_0}^{\mathbf{A}}(\pi_0^w, a_0, \dots, a_{|w|-1}), \dots, \\ &\quad \xi_{u,w,w_{|w|-1}}^{\mathbf{A}}(\pi_{|w|-1}^w, a_0, \dots, a_{|w|-1})), \\ \circ_{u,x,w}^{F_{h,b}(\mathbf{A})}(a, b) &= (\xi_{u,x,w_0}^{\mathbf{A}}(b_0, a_0, \dots, a_{|x|-1}), \dots, \\ &\quad \xi_{u,x,w_{|w|-1}}^{\mathbf{A}}(b_{|w|-1}, a_0, \dots, a_{|x|-1})); \end{aligned}$$

and which to a morphism $f: \mathbf{A} \longrightarrow \mathbf{B}$ of Hall algebras assigns the morphism $F_{h,b}(f) = ((f_w)_u)_{(w,u) \in (S^*)^2}$ from $F_{h,b}(\mathbf{A})$ to $F_{h,b}(\mathbf{B})$ defined, for $(a_0, \dots, a_{|u|-1})$ in $(A_w)_u$, as

$$(a_0, \dots, a_{|u|-1}) \longmapsto (f_{w,u_0}(a_0), \dots, f_{w,u_{|u|-1}}(a_{|u|-1})).$$

The quasi-inverse equivalence from $\mathbf{Alg}(\mathbf{B}_S)$ to $\mathbf{Alg}(\mathbf{H}_S)$ is the functor $F_{b,h}$ which to a Bénabou algebra \mathbf{A} assigns the Hall algebra $F_{b,h}(\mathbf{A})$ that has

- (1) As underlying $S^* \times S$ -sorted set $(A_{w,(s)})_{(w,s) \in S^* \times S}$, and
- (2) As structure of Hall algebra on $(A_{w,(s)})_{(w,s) \in S^* \times S}$ that defined as

$$\begin{aligned} (\pi_i^w)^{F_{b,h}(\mathbf{A})} &= (\pi_i^w)^{\mathbf{A}}, \\ \xi_{u,w,s}^{F_{b,h}(\mathbf{A})}(a, a_0, \dots, a_{|w|-1}) &= a \circ_{u,w,s} \langle a_0, \dots, a_{|w|-1} \rangle_{u,w}; \end{aligned}$$

and which to a homomorphism $f: \mathbf{A} \longrightarrow \mathbf{B}$ of Bénabou algebras assigns the bi-restriction of f to $F_{b,h}(\mathbf{A})$ and $F_{b,h}(\mathbf{B})$.

Next, for a Bénabou algebra \mathbf{A} , we prove that \mathbf{A} and $F_{h,b}(F_{b,h}(\mathbf{A}))$ are isomorphic. Let $f: \mathbf{A} \longrightarrow F_{h,b}(F_{b,h}(\mathbf{A}))$ be the $S^* \times S^*$ -sorted mapping defined, for $(u, w) \in S^* \times S^*$ and $a \in A_{u,w}$, as

$$a \mapsto ((\pi_0^w)^{\mathbf{A}} \circ a, \dots, (\pi_{|w|-1}^w)^{\mathbf{A}} \circ a).$$

The definition is sound because, for $a \in A_{u,w}$, we have that $(\pi_i^w)^{\mathbf{A}} \circ a \in F_{b,h}(A)_{u,w_i}$, hence $((\pi_0^w)^{\mathbf{A}} \circ a, \dots, (\pi_{|w|-1}^w)^{\mathbf{A}} \circ a) \in F_{h,b}(F_{b,h}(A))_{u,w}$. Thus defined f is a homomorphism, since we have, on the one hand, that

$$\begin{aligned} f((\pi_i^w)^{\mathbf{A}}) &= (\pi_0^{(w_i)} \circ \pi_i^w) \\ &= (\langle \pi_0^{(w_i)} \rangle_{(w_i), (w_i)} \circ \pi_i^w) && \text{(by B}_4\text{)} \\ &= (\langle \pi_0^{(w_i)} \circ \langle \pi_0^{(w_i)} \rangle \circ \pi_i^w \rangle_{w, (w_i)}) && \text{(by B}_3\text{)} \\ &= (\langle \pi_0^{(w_i)} \circ \pi_i^w \rangle_{w, (w_i)}) && \text{(by B}_2\text{ and B}_5\text{)} \\ &= (\pi_i^w) && \text{(by B}_3\text{)} \\ &= (\pi_i^w)^{F_{h,b}(F_{b,h}(\mathbf{A}))}, \end{aligned}$$

on the other hand, that

$$\begin{aligned} f(\langle a_0, \dots, a_{|w|-1} \rangle_{u,w}^{\mathbf{A}}) &= ((\pi_0^w)^{\mathbf{A}} \circ \langle a_0, \dots, a_{|w|-1} \rangle_{u,w}^{\mathbf{A}}, \dots, \\ &\quad (\pi_{|w|-1}^w)^{\mathbf{A}} \circ \langle a_0, \dots, a_{|w|-1} \rangle_{u,w}^{\mathbf{A}}) \\ &= (\xi^{F_{b,h}(\mathbf{A})}((\pi_0^w)^{F_{b,h}(\mathbf{A})}, a_0, \dots, a_{|w|-1}), \dots, \\ &\quad \xi^{F_{b,h}(\mathbf{A})}((\pi_{|w|-1}^w)^{F_{b,h}(\mathbf{A})}, a_0, \dots, a_{|w|-1})) \\ &= (\langle a_0, \dots, a_{|w|-1} \rangle_{u,w}^{F_{h,b}(F_{b,h}(\mathbf{A}))}) \\ &= \langle f(a_0), \dots, f(a_{|w|-1}) \rangle_{u,w}^{F_{h,b}(F_{b,h}(\mathbf{A}))}, \end{aligned}$$

and, lastly, that

$$\begin{aligned} f(b \circ^{\mathbf{A}} a) &= ((\pi_0^w)^{\mathbf{A}} \circ (b \circ a), \dots, (\pi_{|w|-1}^w)^{\mathbf{A}} \circ (b \circ a)) \\ &= ((\pi_0^w)^{\mathbf{A}} \circ b \circ \langle a_0, \dots, a_{|w|-1} \rangle, \dots, \\ &\quad (\pi_{|w|-1}^w)^{\mathbf{A}} \circ b \circ \langle a_0, \dots, a_{|w|-1} \rangle) \\ &= (f(b_0) \circ \langle a_0, \dots, a_{|w|-1} \rangle, \dots, \\ &\quad b(b_{|w|-1}) \circ \langle a_0, \dots, a_{|w|-1} \rangle) \\ &= (\xi^{F_{b,h}(\mathbf{A})}(f(b_0), f(a_0), \dots, f(a_{|w|-1})), \dots, \\ &\quad \xi^{F_{b,h}(\mathbf{A})}(f(b_{|w|-1}), f(a_0), \dots, f(a_{|w|-1}))) \\ &= f(b) \circ^{F_{h,b}(F_{b,h}(\mathbf{A}))} f(a). \end{aligned}$$

Reciprocally, let $g: F_{h,b}(F_{b,h}(\mathbf{A})) \longrightarrow \mathbf{A}$ be the $S^* \times S^*$ -sorted mapping defined, for $(u, w) \in S^* \times S^*$ and $b \in F_{h,b}(F_{b,h}(A))$, as

$$b \mapsto \langle b_0, \dots, b_{|w|-1} \rangle_{u,w}^{\mathbf{A}}.$$

The definition is sound because, for $b = (b_0, \dots, b_{|w|-1}) \in F_{h,b}(F_{b,h}(A))$, we have that $b_i \in F_{b,h}(A)_{u,w_i}$, hence $b_i \in A_{u,(w_i)}$, therefore $\langle b_0, \dots, b_{|w|-1} \rangle_{u,w}^{\mathbf{A}} \in A_{u,w}$. Thus defined it is easy to prove that g is a homomorphism.

Now we prove that the homomorphisms f and g are such that $g \circ f = \text{id}_{\mathbf{A}}$ and $f \circ g = \text{id}_{F_{h,b}(F_{b,h}(\mathbf{A}))}$. On the one hand, if $a \in A_{u,w}$, then, by B₃, we have that

$$\langle (\pi_0^w)^{\mathbf{A}} \circ a, \dots, (\pi_{|w|-1}^w)^{\mathbf{A}} \circ a \rangle = a,$$

hence $g \circ f = \text{id}_{\mathbf{A}}$. On the other hand, if $b \in F_{h,b}(F_{b,h}(A))$, then $g_{u,w}$ sends b to $\langle b_0, \dots, b_{|w|-1} \rangle_{u,w}^{\mathbf{A}}$, and $f_{u,w}$ sends $\langle b_0, \dots, b_{|w|-1} \rangle_{u,w}^{\mathbf{A}}$ to

$$((\pi_0^w)^{F_{h,b}(F_{b,h}(\mathbf{A}))}) \circ \langle b_0, \dots, b_{|w|-1} \rangle_{u,w}^{\mathbf{A}}, \dots, (\pi_{|w|-1}^w)^{F_{h,b}(F_{b,h}(\mathbf{A}))}) \circ \langle b_0, \dots, b_{|w|-1} \rangle_{u,w}^{\mathbf{A}}),$$

but this last coincides with

$$((\pi_0^w)^{F_{h,b}(\mathbf{A})}) \circ \langle b_0, \dots, b_{|w|-1} \rangle_{u,w}^{\mathbf{A}}, \dots, (\pi_{|w|-1}^w)^{F_{h,b}(\mathbf{A})}) \circ \langle b_0, \dots, b_{|w|-1} \rangle_{u,w}^{\mathbf{A}},$$

thus, by the axiom \mathbf{B}_1 , we have that this, in its turn, coincides with

$$\langle b_0, \dots, b_{|w|-1} \rangle_{u,w}^{\mathbf{A}},$$

therefore $f_{u,w} \circ g_{u,w}(b) = b$. From which we can assert that $f \circ g = \text{id}_{F_{h,b}(F_{b,h}(\mathbf{A}))}$.

Finally, for a Hall algebra \mathbf{A} we have that \mathbf{A} and $F_{b,h}(F_{h,b}(\mathbf{A}))$ are identical, because $a \in A_{w,s}$ iff $a \in F_{h,b}(A)_{w,(s)}$ iff $a \in F_{b,h}F_{h,b}(A)_{w,s}$. \square

In the following proposition, for a set of sorts S , we state some relations among the just proved equivalence between the categories $\mathbf{Alg}(\mathbf{H}_S)$ and $\mathbf{Alg}(\mathbf{B}_S)$, the adjunctions $\mathbf{T}_{\mathbf{H}_S} \dashv \mathbf{G}_{\mathbf{H}_S}$ and $\mathbf{T}_{\mathbf{B}_S} \dashv \mathbf{G}_{\mathbf{B}_S}$, and the adjunction $\coprod_{1 \times \check{Q}_S} \dashv \Delta_{1 \times \check{Q}_S}$ determined by the mapping $1 \times \check{Q}_S$ from $S^* \times S$ to $S^* \times S^*$ which sends a pair (w, s) in $S^* \times S$ to the pair $(w, (s))$ in $S^* \times S^*$. From these relations we will get as an easy, but interesting, corollary, that, for every $S^* \times S$ -sorted set Σ , $\mathbf{T}_{\mathbf{B}_S}(\coprod_{1 \times \check{Q}_S} \Sigma)$, the free Bénabou algebra on $\coprod_{1 \times \check{Q}_S} \Sigma$, is isomorphic to $\mathbf{BTer}_S(\Sigma)$.

Proposition 46. *Let S be a set of sorts. Then for the diagram*

$$\begin{array}{ccc} \mathbf{Set}^{S^* \times S} & \begin{array}{c} \xleftarrow{\mathbf{G}_{\mathbf{H}_S}} \\ \top \\ \xrightarrow{\mathbf{T}_{\mathbf{H}_S}} \end{array} & \mathbf{Alg}(\mathbf{H}_S) \\ \coprod_{1 \times \check{Q}_S} \downarrow \dashv \uparrow \Delta_{1 \times \check{Q}_S} & & \downarrow \uparrow F_{h,b} \equiv F_{b,h} \\ \mathbf{Set}^{S^* \times S^*} & \begin{array}{c} \xleftarrow{\mathbf{G}_{\mathbf{B}_S}} \\ \top \\ \xrightarrow{\mathbf{T}_{\mathbf{B}_S}} \end{array} & \mathbf{Alg}(\mathbf{B}_S) \end{array}$$

we have that $\Delta_{1 \times \check{Q}_S} \circ \mathbf{G}_{\mathbf{B}_S} = \mathbf{G}_{\mathbf{H}_S} \circ F_{b,h}$ and $\mathbf{T}_{\mathbf{B}_S} \circ \coprod_{1 \times \check{Q}_S} \cong F_{h,b} \circ \mathbf{T}_{\mathbf{H}_S}$.

Proof. The equality $\Delta_{1 \times \check{Q}_S} \circ \mathbf{G}_{\mathbf{B}_S} = \mathbf{G}_{\mathbf{H}_S} \circ F_{b,h}$ follows from the definitions of the functors involved. Then, being $\mathbf{T}_{\mathbf{B}_S} \circ \coprod_{1 \times \check{Q}_S}$ and $F_{h,b} \circ \mathbf{T}_{\mathbf{H}_S}$ left adjoints to the same functor, we can assert that $\mathbf{T}_{\mathbf{B}_S} \circ \coprod_{1 \times \check{Q}_S} \cong F_{h,b} \circ \mathbf{T}_{\mathbf{H}_S}$. \square

Corollary 12. *Let Σ be an S -sorted signature. Then the free Bénabou algebra $\mathbf{T}_{\mathbf{B}_S}(\coprod_{1 \times \check{Q}_S} \Sigma)$ on $\coprod_{1 \times \check{Q}_S} \Sigma$ is isomorphic to the Bénabou algebra $\mathbf{BTer}_S(\Sigma)$ for (S, Σ) .*

Proof. It follows after $\mathbf{BTer}_S(\Sigma) = F_{h,b}(\mathbf{HTer}_S(\Sigma))$. \square

This corollary is interesting because it enables us, on the one hand, to get a more tractable description of the terms in $\mathbf{T}_{\mathbf{B}_S}(\coprod_{1 \times \check{Q}_S} \Sigma)$, and, on the other hand, to give, in the fifth section, an alternative, but equivalent, definition of the concept of morphism of Fujiwara between signatures.

5. MORPHISMS OF FUJIWARA.

In Mathematics it is standard to compare pairs of objects by means of homomorphisms, i.e., mappings from one of them to the other which relate, in a predetermined way, the primitive operations on the source object to the corresponding primitive operations on the target object. But there are natural examples of comparisons between objects, e.g., the derivations in ring theory (see [36], pp. 169–172), which can only be stated by relating the primitive operations on the source object to corresponding (*families of*) *derived* operations on the target object, thus showing the necessity of conveniently generalizing the ordinary concept of homomorphism. Related to this, Fujiwara, in [21], proposed a theory of mappings between algebraic systems, of not necessarily the same similarity type, under which falls the classical concept of homomorphism, but also the above mentioned derivations in ring theory, among others.

Before we outline, briefly, the theory developed by Fujiwara in [21], we agree that for a natural number $n \in \mathbb{N}$ and a standard infinite countable set of variables $V = \{v_n \mid n \in \mathbb{N}\}$, $\downarrow v_n$ is the set $\{v_i \mid i \in n\}$ of the first n variables in V .

Fujiwara defines (in [21], p. 155) for two single-sorted signatures $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ and $\Lambda = (\Lambda_n)_{n \in \mathbb{N}}$, a morphism (that he calls *family of basic mapping-formulas*) from Σ to Λ as, essentially, a pair (Φ, P) , where $\Phi = \{\varphi_\mu \mid \mu \in p\}$ is a set of mapping variables, to be replaced by mappings from a Σ -algebra to another Σ -algebra derived from a Λ -algebra, and $P = (P^n)_{n \in \mathbb{N}}$ a family of mappings such that, for every natural number $n \in \mathbb{N}$, it is the case that

$$P^n \begin{cases} \Phi \times \Sigma_n \longrightarrow T_\Lambda(\Phi \times \downarrow v_n) \\ (\varphi_\mu, \sigma) \longmapsto P_{\varphi_\mu, \sigma}^n \end{cases}$$

i.e., P^n sends a pair (φ_μ, σ) , with φ_μ a mapping variable and σ a formal n -ary operation, to a term $P_{\varphi_\mu, \sigma}^n$ for Λ on the set of variables $\Phi \times \downarrow v_n$. To shorten the notation we identify the variables (φ_μ, v_i) in $\Phi \times \downarrow v_n$ to the expressions $\varphi_\mu(v_i)$, for $\mu \in p$ and $i \in n$. We remark that for Fujiwara the cardinal number of a set of mapping variables is not necessarily finite. However, for us, leaving out some examples shown below, and in order not to complicate still more the notation, they will be finite sets.

Let (Φ, P) be a morphism from Σ to Λ , \mathbf{B} a Λ -algebra, $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$. Then Fujiwara (in [21], p. 155) assigns to the formal n -ary operation σ the n -ary operation $F_\sigma^{\mathbf{B}^\Phi}$ from $(B^\Phi)^n$ to B^Φ obtained as the composition of the vertical mappings in the following diagram

$$\begin{array}{ccc} & (B^\Phi)^n & \\ & \downarrow & \\ F_\sigma^{\mathbf{B}^\Phi} & B^{\Phi \times \downarrow v_n} & \\ & \downarrow & \\ & \langle P_{\varphi_\mu, \sigma}^n \rangle_{\mu \in p} & \\ & \downarrow & \\ & B^\Phi & \xrightarrow{\text{pr}_\mu} B \end{array}$$

where we have that

- (1) For every $\mu \in p$, pr_μ is the canonical projection from B^Φ to B ,
- (2) For every $\mu \in p$, $P_{\varphi_\mu, \sigma}^n$ is the term operation on the Λ -algebra \mathbf{B} determined by the term $P_{\varphi_\mu, \sigma}^n \in T_\Lambda(\Phi \times \downarrow v_n)$,

- (3) The vertical mapping from $(B^\Phi)^n$ to $B^{\Phi \times \downarrow v_n}$ is the canonical isomorphism between both, and
 (4) $\langle P_{\varphi_\mu, \sigma}^{n, \mathbf{B}} \rangle_{\mu \in p}$ is the unique mapping from $B^{\Phi \times \downarrow v_n}$ to B^Φ such that, for every $\mu \in p$, $\text{pr}_\mu \circ \langle P_{\varphi_\mu, \sigma}^{n, \mathbf{B}} \rangle_{\mu \in p} = P_{\varphi_\mu, \sigma}^{n, \mathbf{B}}$.

In this way, Fujiwara (in [21], p. 155) associates to every Λ -algebra \mathbf{B} a corresponding Σ -algebra $P(\mathbf{B})$ with B^Φ as underlying set and $(F_\sigma^{\mathbf{B}^\Phi})_{\sigma \in \bigcup_{n \in \mathbb{N}} \Sigma_n}$ as family of structural operations. However he does not extend this association up to a functor from the category $\mathbf{Alg}(\Lambda)$ to the category $\mathbf{Alg}(\Sigma)$ (probably reflecting that the time was not still ripe for a generalized acceptance of the category-theoretical way of thinking in mathematics).

For a morphism (Φ, P) from Σ to Λ , a Σ -algebra \mathbf{A} , and a Λ -algebra \mathbf{B} , the association just recalled led Fujiwara to take (in [21], pp. 155–156, through the so-called *algebraic Taylor's expansion theorem*), as (Φ, P) -mappings from \mathbf{A} to \mathbf{B} those families of mappings $(f_\mu)_{\mu \in p} \in \prod_{\mu \in p} \text{Hom}(A, B)$ such that the mapping $\langle f_\mu \rangle_{\mu \in p}$ from A to B^Φ is a Σ -homomorphism from the Σ -algebra \mathbf{A} to the Σ -algebra $P(\mathbf{B})$, or, what is equivalent, those families $(f_\mu)_{\mu \in p}$ such that, for every $n \in \mathbb{N}$, every $\sigma \in \Sigma_n$ and any elements a_0, \dots, a_{n-1} in A , all the identities obtained by substituting $(f_\mu)_{\mu \in p}$, $F_\sigma^{\mathbf{A}}$, $P_{\varphi_\mu, \sigma}^{n, \mathbf{B}}$, and a_0, \dots, a_{n-1} for $(\varphi_\mu)_{\mu \in p}$, σ , $P_{\varphi_\mu, \sigma}^n$, and v_0, \dots, v_{n-1} , respectively, in all formal equations

$$\varphi_\mu(\sigma(v_0, \dots, v_{n-1})) = P_{\varphi_\mu, \sigma}^n \begin{pmatrix} \varphi_0(v_0), & \dots, & \varphi_0(v_{n-1}) \\ \dots\dots\dots\dots\dots\dots \\ \varphi_\mu(v_0), & \dots, & \varphi_\mu(v_{n-1}) \\ \dots\dots\dots\dots\dots\dots \\ \varphi_{p-1}(v_0), & \dots, & \varphi_{p-1}(v_{n-1}) \end{pmatrix}$$

are true in \mathbf{B} .

Example. Let $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ be a single-sorted signature, $\Phi = \{ \varphi_\mu \mid \mu \in m \}$, with $m \in \mathbb{N}$, and $P = (P^n)_{n \in \mathbb{N}}$ the family defined, for every natural number $n \in \mathbb{N}$, as follows

$$P^n \left\{ \begin{array}{l} \Phi \times \Sigma_n \longrightarrow T_\Sigma(\Phi \times \downarrow v_n) \\ (\varphi_\mu, \sigma) \longmapsto \sigma(\varphi_\mu(v_0), \dots, \varphi_\mu(v_{n-1})) \end{array} \right.$$

Then, for a Σ -algebra \mathbf{B} , $P(\mathbf{B})$ is \mathbf{B}^m , the direct m -power of \mathbf{B} .

Example. Let $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ be a single-sorted signature such that $\Sigma_2 = \{ +, -, \cdot \}$ and $\Sigma_n = \emptyset$, if $n \neq 2$, $\Phi = \{ \varphi_\mu \mid \mu \in \mathbb{N} \}$, and $P = (P^n)_{n \in \mathbb{N}}$ the family defined, for $n \neq 2$, as the unique mapping from \emptyset to $T_\Sigma(\Phi \times \downarrow v_n)$, and, for $n = 2$, as follows

- (1) $P_{\varphi_\mu, +}^2 = \varphi_\mu(v_0) + \varphi_\mu(v_1)$.
- (2) $P_{\varphi_\mu, -}^2 = \varphi_\mu(v_0) - \varphi_\mu(v_1)$.
- (3) $P_{\varphi_\mu, \cdot}^2 = \sum_{i=0}^\mu \varphi_{\mu-i}(v_0) \cdot \varphi_\mu(v_1)$.

Then, for a ring \mathbf{B} , $P(\mathbf{B})$ can be considered as the ring of formal power series over \mathbf{B} .

Example. Let $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ be a single-sorted signature such that $\Sigma_2 = \{ +, -, \cdot \}$ and $\Sigma_n = \emptyset$, if $n \neq 2$, $\Phi = \{ \varphi_{\mu, \nu} \mid (\mu, \nu) \in m \times m \}$, with $m \in \mathbb{N}$, and $P = (P^n)_{n \in \mathbb{N}}$ the family defined, for $n \neq 2$, as the unique mapping from \emptyset to $T_\Sigma(\Phi \times \downarrow v_n)$, and, for $n = 2$, as follows

- (1) $P_{\varphi_{\mu, \nu}, +}^2 = \varphi_{\mu, \nu}(v_0) + \varphi_{\mu, \nu}(v_1)$.
- (2) $P_{\varphi_{\mu, \nu}, -}^2 = \varphi_{\mu, \nu}(v_0) - \varphi_{\mu, \nu}(v_1)$.
- (3) $P_{\varphi_{\mu, \nu}, \cdot}^2 = \sum_{\lambda=0}^{m-1} \varphi_{\mu, \lambda}(v_0) \cdot \varphi_{\lambda, \nu}(v_1)$.

Then, for a ring \mathbf{B} , $P(\mathbf{B})$ can be considered as the matrix ring of degree m over \mathbf{B} .

We point out that under this example falls the concept of derivation from a ring into a like one. We recall that, for a pair of rings \mathbf{A} and \mathbf{B} and two ring homomorphisms f and g from \mathbf{A} to \mathbf{B} , a mapping d from A to B is called an (f, g) -derivation from \mathbf{A} to \mathbf{B} iff, for every $x, y \in A$, we have that

- (1) $d(x + y) = d(x) + d(y)$.
- (2) $d(xy) = d(x)f(y) + g(x)d(y)$.

Thus defined d is not a ring homomorphism from \mathbf{A} to \mathbf{B} , however, if d is any (f, g) -derivation from \mathbf{A} to \mathbf{B} , then the matrix $\begin{pmatrix} f & \kappa_0 \\ d & g \end{pmatrix}$, where κ_0 is the mapping from A to B that is constantly 0, defines a ring homomorphism from the ring \mathbf{A} to $P(\mathbf{B}) = \mathbf{B}^{2 \times 2}$, the matrix ring of degree 2 over \mathbf{B} , i.e., the matrix is a (Φ, P) -mapping from \mathbf{A} to \mathbf{B} .

Example. Let $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ be a single-sorted signature such that $\Sigma_2 = \{+, -, \cdot\}$ and $\Sigma_n = \emptyset$, if $n \neq 2$, $\Lambda = (\Lambda_n)_{n \in \mathbb{N}}$ a single-sorted signature such that $\Lambda_2 = \{+', -', \cdot'\}$ and $\Lambda_n = \emptyset$, if $n \neq 2$, $\Phi = \{\varphi\}$, and $P = (P^n)_{n \in \mathbb{N}}$ the family defined, for $n \neq 2$, as the unique mapping from \emptyset to $T_\Lambda(\Phi \times \downarrow v_n)$, and, for $n = 2$, as follows

- (1) $P_{\varphi, +}^2 = \varphi(v_0) +' \varphi(v_1)$.
- (2) $P_{\varphi, -}^2 = \varphi(v_0) -' \varphi(v_1)$.
- (3) $P_{\varphi, \cdot}^2 = \varphi(v_0) \cdot' \varphi(v_1) -' \varphi(v_1) \cdot' \varphi(v_0)$.

Then, for a ring \mathbf{B} , $P(\mathbf{B})$ is a Lie ring.

After this Fujiwara proceeds, among other things, to define the composition of morphisms between single-sorted signatures. But for this he introduces (in [21], pp. 157–160) a certain notation that we explain as follows. Given a morphism $(\Phi, P): \Sigma \longrightarrow \Lambda$, an index $\mu \in p$, and a set X , we denote by $F_{\varphi_\mu}^{(\Phi, P), X}$ the Σ -homomorphism from $\mathbf{T}_\Sigma(X)$ to $\mathbf{T}_\Lambda(\Phi \times X)$ obtained as the composition of the vertical mappings in the following diagram

$$\begin{array}{ccc}
 X & \xrightarrow{\eta_X} & \mathbf{T}_\Sigma(X) \\
 & \searrow \eta_{\Phi \times X}^b & \downarrow (\eta_{\Phi \times X}^b)^\sharp \\
 & & \mathbf{T}_\Lambda(\Phi \times X)^\Phi \\
 & & \downarrow \text{pr}_\mu \\
 & & \mathbf{T}_\Lambda(\Phi \times X)
 \end{array}
 \quad
 \begin{array}{l}
 \longleftarrow \\
 \longleftarrow \\
 \longleftarrow \\
 \longleftarrow \\
 \longleftarrow \\
 \longleftarrow
 \end{array}
 F_{\varphi_\mu}^{(\Phi, P), X}$$

where $\eta_{\Phi \times X}^b: X \longrightarrow \mathbf{T}_\Lambda(\Phi \times X)^\Phi$ is the adjunct to $\eta_{\Phi \times X}: \Phi \times X \longrightarrow \mathbf{T}_\Lambda(\Phi \times X)$, and pr_μ the μ -th projection from $\mathbf{T}_\Lambda(\Phi \times X)^\Phi$ to $\mathbf{T}_\Lambda(\Phi \times X)$.

Now given $(\Phi, P): \Sigma \longrightarrow \Lambda$ and $(\Psi, Q): \Lambda \longrightarrow \Omega$, with $\Phi = \{\varphi_\mu \mid \mu \in p\}$ and $\Psi = \{\psi_\nu \mid \nu \in q\}$, Fujiwara defines (in [21], p. 164) the composition of (Φ, P) and (Ψ, Q) as the pair $(\Psi \times \Phi, Q \circ P)$, where, for every $n \in \mathbb{N}$, $(Q \circ P)^n$ is the mapping

$$(Q \circ P)^n \left\{ \begin{array}{l}
 (\Psi \times \Phi) \times \Sigma_n \longrightarrow \mathbf{T}_\Omega((\Psi \times \Phi) \times \downarrow v_n) \\
 ((\psi_\nu, \varphi_\mu), \sigma) \longmapsto (Q \circ P)_{(\psi_\nu, \varphi_\mu), \sigma}^n = F_{\psi_\nu}^{(\Psi, Q), \Phi \times \downarrow v_n}(P_{\varphi_\mu, \sigma}^n)
 \end{array} \right.$$

from this he obtains (in [21], pp. 165–166), for every Ω -algebra \mathbf{C} , an isomorphism between the Σ -algebras $Q(P(\mathbf{C}))$ and $Q \circ P(\mathbf{C})$. However he does not consider in [21] neither the corresponding category with objects the single-sorted signatures, morphisms between single-sorted signatures the families of basic mapping-formulas, and composition the just defined composition of morphisms, nor, obviously, the

contravariant pseudo-functor defined on such a category of signatures, although the essential components of the subject matter are there (and patiently waiting).

In this section, following the work by Fujiwara in [21], we generalize the morphisms in \mathbf{Sig} in such a way that a signature morphism from a signature into a like one, to be called from now on a *morphism of Fujiwara*, or more briefly, a *polyderivator*, will consist of two suitably related mappings: On the one hand, a mapping which relates the sets of sorts of the signatures and assigns to each sort in the first, a derived sort in the second, i.e., a word on the set of sorts of the second, and, on the other hand, a mapping which assigns to each formal operation in the first, a family of terms in the second, all in such a way that both transformations are compatible. This type of signature morphism, from which we will get a category \mathbf{Sig}_{pd} , with the same objects that \mathbf{Sig} , will allow us to generalize, concordantly, the morphisms between algebras.

We will also prove that the category \mathbf{Sig}_{pd} of signatures and polyderivators is isomorphic to the Kleisli category for a monad in \mathbf{Sig} , and that fact will confirm, to some extent, the naturalness of the concept of polyderivator. Furthermore, the contravariant functor $\mathbf{Alg}: \mathbf{Sig} \rightarrow \mathbf{Cat}$ will be lifted up to a contravariant pseudo-functor $\mathbf{Alg}_{\text{pd}}: \mathbf{Sig}_{\text{pd}} \rightarrow \mathbf{Cat}$ and, by applying, once more, the construction of Ehresmann-Grothendieck, we will get the category \mathbf{Alg}_{pd} of algebras and algebra morphisms that will have the polyderivators as a component. In the same way, the pseudo-functor $\mathbf{Ter}: \mathbf{Sig} \rightarrow \mathbf{Cat}$ will be lifted up to a pseudo-functor $\mathbf{Ter}_{\text{pd}}: \mathbf{Sig}_{\text{pd}} \rightarrow \mathbf{Cat}$.

On the other hand, to account exactly for the invariant character of the realization of terms in algebras under the polyderivators, we prove the existence of a pseudo-extranatural transformation (Tr, θ) from a pseudo-functor $\mathbf{Alg}_{\text{pd}}(\cdot) \times \mathbf{Ter}_{\text{pd}}(\cdot)$, on $\mathbf{Sig}_{\text{pd}}^{\text{op}} \times \mathbf{Sig}_{\text{pd}}$ to \mathbf{Cat} , to the functor \mathbf{K}_{Set} , between the same categories, that is constantly \mathbf{Set} .

Before we define the polyderivators, we agree that $\mathbb{T}_\star = (\mathbb{T}_\star, \check{\jmath}, \lambda)$ is the standard monad in \mathbf{Set} for the monoid specification, where, for every set S , $\mathbb{T}_\star(S) = S^\star$ is the set $\bigcup_{n \in \mathbb{N}} S^n$, $\check{\jmath}_S: S \rightarrow S^\star$ the inclusion of S into S^\star , and $\lambda_S: S^{\star\star} \rightarrow S^\star$ the merging of strings of words to words. To simplify the notation, we will write (s) instead of $\check{\jmath}_S(s)$. Furthermore, if $\varphi: S \rightarrow T^\star$, then $\varphi^\sharp: S^\star \rightarrow T^\star$ is the underlying mapping of the canonical extension of φ to the free monoid $\mathbb{T}_\star(S)$ on S and φ^\star the unique monoid homomorphism from $\mathbb{T}_\star(S)$ to $\mathbb{T}_\star(T^\star)$, the free monoid on the underlying set of the free monoid on T , such that $\varphi^\star \circ \check{\jmath}_S = \check{\jmath}_{T^\star} \circ \varphi$.

Definition 28. Let Σ and Λ be signatures. A *polyderivator* from Σ to Λ is a pair $\mathbf{d} = (\varphi, d)$, where $\varphi: S \rightarrow T^\star$ while $d: \Sigma \rightarrow \mathbf{BTer}_T(\Lambda)_{\varphi^\sharp \times \varphi}$.

Therefore, if $\mathbf{d}: \Sigma \rightarrow \Lambda$ is a polyderivator, then, for every $(w, s) \in S^\star \times S$, we have that

$$d_{w,s}: \Sigma_{w,s} \rightarrow \mathbf{BTer}_T(\Lambda)_{\varphi^\sharp(w), \varphi(s)} = \mathbb{T}_\Lambda(\downarrow \varphi^\sharp(w))_{\varphi(s)},$$

and, given that $\Delta_{\varphi^\sharp \times \varphi} = \Delta_{1 \times \check{\jmath}_S} \circ \Delta_{\varphi^\sharp \times \varphi^\sharp}$ and the functor $\prod_{1 \times \check{\jmath}_S}$ is left adjoint to the functor $\Delta_{1 \times \check{\jmath}_S}$, d is, essentially, an $S^\star \times S^\star$ -sorted mapping

$$\theta^{1 \times \check{\jmath}_S}(d): \prod_{1 \times \check{\jmath}_S} \Sigma \rightarrow \mathbf{BTer}_T(\Lambda)_{\varphi^\sharp \times \varphi^\sharp}.$$

From now on, for every polyderivator \mathbf{d} , we identify d and $\theta^{1 \times \check{\jmath}_S}(d)$.

Remark. For every word w in S^\star , $\varphi^\sharp(w) = \varphi(w_0) \lambda \cdots \lambda \varphi(w_{|w|-1})$ is a word on T and if we agree that, for every $\alpha \in |w|$, $p_\alpha = |\varphi(w_\alpha)|$, then, for every $\alpha \in |w|$ and $i \in \sum_{\alpha \in |w|} p_\alpha$, we have that

$$\varphi^\sharp(w)_i = \varphi(w_\alpha)_{i - \sum_{\beta \in \alpha} p_\beta} \quad \text{iff} \quad \sum_{\beta \in \alpha} p_\beta \leq i \leq \sum_{\beta \in \alpha+1} p_\beta - 1.$$

Hence $\varphi^\sharp(w)$ has the configuration

$$\varphi^\sharp(w) = \overbrace{(\varphi^\sharp(w)_0, \dots, \varphi^\sharp(w)_{p_0-1}, \dots)}^{\varphi(w_0)} \\ \dots, \overbrace{(\varphi^\sharp(w)_{\sum_{\beta \in \alpha} p_\beta}, \dots, \varphi^\sharp(w)_{\sum_{\beta \in \alpha+1} p_\beta-1}, \dots)}^{\varphi(w_\alpha)} \\ \dots, \overbrace{(\varphi^\sharp(w)_{\sum_{\beta \in |w|-1} p_\beta}, \dots, \varphi^\sharp(w)_{\sum_{\beta \in |w|} p_\beta-1})}^{\varphi(w_{|w|-1})}.$$

Then, for a formal operation $\sigma: w \longrightarrow s$ in Σ , it is useful to think about the family of terms $d_{w,s}(\sigma) \in \mathbf{T}_\Lambda(\downarrow \varphi^\sharp(w))_{\varphi(s)}$, denoted by $d(\sigma): \varphi^\sharp(w) \longrightarrow \varphi(s)$, as being graphically represented by

$$d(\sigma) : \begin{pmatrix} \varphi^\sharp(w)_0 & \cdots & \varphi^\sharp(w)_{p_0-1} \\ \vdots & \ddots & \vdots \\ \varphi^\sharp(w)_{\sum_{\beta \in |w|-1} p_\beta} & \cdots & \varphi^\sharp(w)_{\sum_{\beta \in |w|} p_\beta-1} \end{pmatrix} \longrightarrow (\varphi(s)_0, \dots, \varphi(s)_{|\varphi(s)|-1}).$$

For every signature Λ , $\mathbf{B}\mathbf{Ter}_T(\Lambda)$ is the underlying ms-set of $\mathbf{B}\mathbf{Ter}_T(\Lambda)$, the Bénabou algebra for (T, Λ) , and $\mathbf{B}\mathbf{Ter}_T(\Lambda)$ is isomorphic to $\mathbf{T}_{\mathbf{B}T}(\coprod_{1 \times \check{\mathcal{Q}}_T} \Lambda)$, by Corollary 12, hence the polyderivors can also be defined as pairs $\mathbf{d} = (\varphi, d)$, where $\varphi: S \longrightarrow T^*$ while d is an $S^* \times S$ -sorted mapping from Σ to $\mathbf{T}_{\mathbf{B}T}(\coprod_{1 \times \check{\mathcal{Q}}_T} \Lambda)_{\varphi^\sharp \times \varphi}$, or, equivalently, an $S^* \times S^*$ -sorted mapping from $\coprod_{1 \times \check{\mathcal{Q}}_S} \Sigma$ to $\mathbf{T}_{\mathbf{B}T}(\coprod_{1 \times \check{\mathcal{Q}}_T} \Lambda)_{\varphi^\sharp \times \varphi^\sharp}$.

Example. Let Σ be a signature and $p \in \mathbb{N}$. Then taking

- (1) As $\varphi: S \longrightarrow S^*$ the mapping which sends $s \in S$ to the word $\wedge_{\mu \in p}(s)$ and,
- (2) For $(w, s) \in S^* \times S$, as $d_{w,s}$ the mapping from $\Sigma_{w,s}$ to $\mathbf{T}_\Sigma(\downarrow \varphi^\sharp(w))_s^p$ (because, in this case, $\mathbf{T}_\Sigma(\downarrow \varphi^\sharp(w))_{\varphi(s)} = \mathbf{T}_\Sigma(\downarrow \varphi^\sharp(w))_s^p$), which sends $\sigma \in \Sigma_{w,s}$ to

$$(\sigma(v_0^{w_0}, v_p^{w_1}, \dots, v_{(|w|-1)p}^{w_{|w|-1}}), \dots, \sigma(v_{p-1}^{w_0}, v_{2p-1}^{w_1}, \dots, v_{|w|p-1}^{w_{|w|-1}})),$$

in $\mathbf{T}_\Sigma(\downarrow \varphi^\sharp(w))_s^p$,

we have that $\mathbf{d} = (\varphi, d)$ is a polydivor from Σ into itself. Later on, after having defined the category $\mathbf{Sig}_{p\partial}$, of signatures and polyderivors, and the pseudo-functor $\mathbf{Alg}_{p\partial}$ from $\mathbf{Sig}_{p\partial}$ to \mathbf{Cat} , we will see that, for a signature Σ , a natural number $p \in \mathbb{N}$, and a Σ -algebra \mathbf{B} , the result of the action of $\mathbf{Alg}_{p\partial}$ on \mathbf{B} is \mathbf{B}^p .

For more examples of polyderivors we refer to the last section of this paper, where we consider polyderivors between the (many-sorted) signatures of Hall and Bénabou.

Example. Let $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ and $\Lambda = (\Lambda_n)_{n \in \mathbb{N}}$ be two single-sorted signatures and (Φ, P) , with $\Phi = \{\varphi_\mu \mid \mu \in p\}$, a family of basic mapping-formulas from Σ to Λ as defined by Fujiwara in [21], p. 155. Then by associating

- (1) To the single-sorted signatures Σ and Λ , the signatures $(1, (\Sigma_{n,0})_{(n,0) \in 1^* \times 1})$ and $(1, (\Lambda_{n,0})_{(n,0) \in 1^* \times 1})$, respectively, where, for every $n \in 1^* \cong \mathbb{N}$, $\Sigma_{n,0} = \Sigma_n$ and $\Lambda_{n,0} = \Lambda_n$, and
- (2) To the morphism (Φ, P) the pair (κ_p, d) , where κ_p is the mapping from 1 to 1^* which sends $0 \in 1$ to $p \in 1^*$ and d the $1^* \times 1$ -sorted mapping from $(\Sigma_{n,0})_{(n,0) \in 1^* \times 1}$ to $(\mathbf{T}_\Lambda(\downarrow \kappa_p^\sharp(n))_{\kappa_p(0)})_{(n,0) \in 1^* \times 1} \cong (\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^p)_{n \in \mathbb{N}}$ which, for $n \in 1^*$, sends $\sigma \in \Sigma_n$ to $d_{n,0}(\sigma) = (P_{\varphi_0, \sigma}^n, \dots, P_{\varphi_{p-1}, \sigma}^n)$,

we have that the families of basic mapping-formulas defined by Fujiwara for the single-sorted case fall, obviously, under the concept of polydivor. Consequently,

all the examples we have put in this section before the definition of polyderivator, once reformulated as just said, are also examples of polyderivators.

Example. A standard signature morphism from a signature (S, Σ) into a like one (T, Λ) , as defined in the first section, is a pair of mappings (φ, d) , where $\varphi: S \longrightarrow T$ is a mapping in **Set**, which sends sort symbols to sort symbols, while $d: \Sigma \longrightarrow \Lambda_{\varphi^* \times \varphi}$ is a morphism in **Sig**(S), which sends formal operations to formal operations, respecting the assignment of sorts. But from $\varphi: S \longrightarrow T$ we get $\check{Q}_T \circ \varphi: S \longrightarrow T^*$, and from $d: \Sigma \longrightarrow \Lambda_{\varphi^* \times \varphi}$, because there exists a canonical embedding from $\Lambda_{\varphi^* \times \varphi}$ into $(\coprod_{1 \times \check{Q}_T} \Lambda)_{(\check{Q}_T \circ \varphi)^\# \times (\check{Q}_T \circ \varphi)}$, we get the composite mapping

$$\Sigma \xrightarrow{d} \Lambda_{\varphi^* \times \varphi} \longrightarrow (\coprod_{1 \times \check{Q}_T} \Lambda)_{(\check{Q}_T \circ \varphi)^\# \times (\check{Q}_T \circ \varphi)} \longrightarrow \mathbf{T}_{B_T}(\coprod_{1 \times \check{Q}_T} \Lambda)_{(\check{Q}_T \circ \varphi)^\# \times (\check{Q}_T \circ \varphi)}.$$

In this way we have assigned to every standard signature morphism a corresponding polyderivator from the source to the target signature of the signature morphism. Thus the standard signature morphisms fall under the concept of polyderivator.

Our next goal is to define the composition of polyderivators in order to get the category **Sig**_{pd}, of signatures and polyderivators. To attain the just stated goal we need to recall beforehand the concept of derivator from a signature into a like one as defined by Goguen, Thatcher and Wagner in [27], p. 86, and to set out some of its properties.

Definition 29. Let Σ and Λ be signatures. Then a *derivator from Σ to Λ* is a pair $\mathbf{d} = (\varphi, d)$, with $\varphi: S \longrightarrow T$ and $d: \Sigma \longrightarrow \mathbf{HTer}_T(\Lambda)_{\varphi^* \times \varphi}$.

Therefore, if $\mathbf{d}: \Sigma \longrightarrow \Lambda$ is a derivator, then, for every $(w, s) \in S^* \times S$, we have that

$$d_{w,s}: \Sigma_{w,s} \longrightarrow \mathbf{HTer}_T(\Lambda)_{\varphi^*(w), \varphi(s)} = \mathbf{T}_\Lambda(\downarrow \varphi^*(w))_{\varphi(s)}$$

sends a formal operation $\sigma: w \longrightarrow s$ to a term $d_{w,s}(\sigma): \varphi^*(w) \longrightarrow \varphi(s)$, i.e., a term for Λ of type $(\downarrow \varphi^*(w), \varphi(s))$, and all in such a way that the arities and coarities are preserved, modulus the correspondence between the sorts given by the mapping φ .

For a signature Λ , we have that $\mathbf{HTer}_T(\Lambda)$ is the underlying ms-set of $\mathbf{HTer}_T(\Lambda)$, the Hall algebra for (T, Λ) . But $\mathbf{HTer}_T(\Lambda)$ is isomorphic to $\mathbf{T}_{H_T}(\Lambda)$, the free H_T -algebra on Λ , by Proposition 41. Consequently the derivators can be defined, alternative, but equivalently, as pairs $\mathbf{d} = (\varphi, d)$ with $\varphi: S \longrightarrow T$ and $d: \Sigma \longrightarrow \mathbf{T}_{H_T}(\Lambda)$. Thus, taking into account the equivalence between the categories **Alg**(H_T) and **Alg**(B_T), we can state the following

Corollary 13. *Every derivator is a polyderivator (although, obviously, not every polyderivator is a derivator).*

Remark. For a *single-sorted* signature Σ , a derivator from Σ to Σ , i.e., an endoderivator of Σ , is, essentially, a morphism d in **Set** ^{\mathbb{N}} from Σ to $(\mathbf{T}_\Sigma(\downarrow v_n))_{n \in \mathbb{N}}$. Therefore, for every $n \in \mathbb{N}$, we have that

$$d_n: \Sigma_n \longrightarrow \mathbf{T}_\Sigma(\downarrow v_n)$$

sends a formal operation $\sigma \in \Sigma_n$ to a term $d(\sigma) \in \mathbf{T}_\Sigma(\downarrow v_n)$. From this it follows that the concept of *hypersubstitution* as defined, e.g., by Denecke and Wismath in [15], p. 13 (and itself the main tool from which it is constructed the theory of *hyperidentities*), is a particular case of the concept of derivator and, consequently, also of that of polyderivator.

The following two are examples of derivators that originate in the work by Higman and B.H. Neumann in group theory (see [35]).

Example. Let $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ be a single-sorted signature such that $\Sigma_0 = \{1\}$, $\Sigma_2 = \{/ \}$ and $\Sigma_n = \emptyset$, if $n \neq 0, 2$, $\Lambda = (\Lambda_n)_{n \in \mathbb{N}}$ a single-sorted signature such that $\Lambda_0 = \{1\}$, $\Lambda_1 = \{-^1\}$, $\Lambda_2 = \{\cdot\}$ and $\Lambda_n = \emptyset$, if $n \neq 0, 1, 2$, and $d = (d_n)_{n \in \mathbb{N}}$ the mapping from Σ to Λ whose n -th coordinate mapping is, for $n \neq 0, 2$, the unique mapping from \emptyset to $T_\Lambda(\downarrow v_n)$, and, for $n = 0, 2$, that defined as follows

- (1) $d_0(1) = 1$;
- (2) $d_2(/) = v_0 \cdot v_1^{-1}$.

Then d is a derivor from Σ to Λ . This derivor defines $/$, the “division”, in terms of \cdot , the “multiplication”, and $^{-1}$, the “inverse”.

Example. For the same two single-sorted signatures as in the preceding example let $e = (e_n)_{n \in \mathbb{N}}$ be the mapping from Λ to Σ whose n -th coordinate mapping is, for $n \neq 0, 1, 2$, the unique mapping from \emptyset to $T_\Sigma(\downarrow v_n)$, and, for $n = 0, 1, 2$, that defined as follows

- (1) $e_0(1) = 1$;
- (2) $e_1(-^1) = 1/v_0$;
- (3) $e_2(\cdot) = v_0/(1/v_1)$.

Then e is a derivor from Λ to Σ . This derivor defines \cdot , the “multiplication”, and $^{-1}$, the “inverse”, in terms of $/$, the “division”.

The next two examples of derivors come from the proof by M. H. Stone of the subsumption of the theory of Boolean algebras under the theory of rings (see [56], pp. 43–48).

Example. Let $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ be a single-sorted signature such that $\Sigma_0 = \{0, 1\}$, $\Sigma_1 = \{ '\}$, $\Sigma_2 = \{\wedge, \vee\}$ and $\Sigma_n = \emptyset$, if $n \neq 0, 1, 2$, $\Lambda = (\Lambda_n)_{n \in \mathbb{N}}$ a single-sorted signature such that $\Lambda_0 = \{0, 1\}$, $\Lambda_1 = \{-\}$, $\Lambda_2 = \{\cdot, +\}$ and $\Lambda_n = \emptyset$, if $n \neq 0, 1, 2$, and $d = (d_n)_{n \in \mathbb{N}}$ the mapping from Σ to Λ whose n -th coordinate mapping is, for $n \neq 0, 1, 2$, the unique mapping from \emptyset to $T_\Lambda(\downarrow v_n)$, and, for $n = 0, 1, 2$, that defined as follows

- (1) $d_0(0) = 0$ and $d_0(1) = 1$;
- (2) $d_1(') = 1 + v_0$;
- (3) $d_2(\wedge) = v_0 \cdot v_1$ and $d_2(\vee) = v_0 + v_1 + v_0 \cdot v_1$.

Then d is a derivor from Σ to Λ . This derivor defines the “Boolean algebra” operations in terms of the “Boolean ring” operations.

Example. For the same two single-sorted signatures as in the preceding example let $e = (e_n)_{n \in \mathbb{N}}$ be the mapping from Λ to Σ whose n -th coordinate mapping is, for $n \neq 0, 1, 2$, the unique mapping from \emptyset to $T_\Sigma(\downarrow v_n)$, and, for $n = 0, 1, 2$, that defined as follows

- (1) $e_0(0) = 0$ and $e_0(1) = 1$;
- (2) $e_1(-) = v_0$;
- (3) $e_2(\cdot) = v_0 \wedge v_1$ and $e_2(+)= (v_0 \wedge v_1') \vee (v_0' \wedge v_1)$.

Then e is a derivor from Λ to Σ . This derivor defines the “Boolean ring” operations in terms of the “Boolean algebra” operations.

Another example of derivor is that provided by Gödel (see [23]) in his work about an interpretation of the intuitionistic propositional logic into a modal extension of the classical propositional logic.

Example. Let $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ be a single-sorted signature such that $\Sigma_1 = \{\neg_i\}$, $\Sigma_2 = \{\wedge_i, \vee_i, \rightarrow_i\}$ and $\Sigma_n = \emptyset$, if $n \neq 1, 2$, $\Lambda = (\Lambda_n)_{n \in \mathbb{N}}$ a single-sorted signature such that $\Lambda_1 = \{\neg_c, \square\}$, $\Lambda_2 = \{\wedge_c, \vee_c, \rightarrow_c\}$, and $\Lambda_n = \emptyset$, if $n \neq 1, 2$, and $g = (g_n)_{n \in \mathbb{N}}$ the family defined, for $n \neq 1, 2$, as the unique mapping from \emptyset to $T_\Lambda(\downarrow v_n)$, and, for $n = 1, 2$, as follows

- (1) $g_1(\neg_i) = \neg_c \Box v_0$.
- (2) $g_2(\wedge_i) = v_0 \wedge_c v_1$.
- (3) $g_2(\vee_i) = \Box v_0 \vee_c \Box v_1$.
- (4) $g_2(\rightarrow_i) = \Box v_0 \rightarrow_c \Box v_1$.

Then d is a derivator from Σ to Λ . This derivator defines the “intuitionistic” connectives in terms of the “classical” connectives together with \Box , the operator of “necessity”.

Next we proceed to define the composition of derivators and to prove that the corresponding category of signatures and derivators, denoted by \mathbf{Sig}_δ , is isomorphic to the Kleisli category for a monad \mathbb{T}_δ in \mathbf{Sig} . This last result means, in other words, that the derivators are indiscernible from the morphisms of the Kleisli category for the monad \mathbb{T}_δ in \mathbf{Sig} , thus confirming its mathematical naturalness. By proceeding in this way we, on the one hand, move one step forward, from the standpoint of category theory, in the investigation of some of the most notable positive properties of the category \mathbf{Sig}_δ , with regard to what has been done in [27], and, on the other hand, get a model for the subsequent work we have to do with the polyderivators.

Remark. The derivators, that set up relations between signatures, and the *simple* entailment system morphisms (see [47], p. 297, for the definition of this concept), that do a similar, although more sophisticated, task, but for entailment systems, are related through the following proportion: derivators are to standard signature morphisms as simple entailment system morphisms are to standard (*plain*) entailment system morphisms. Besides, the proportionality arrives, as a matter of fact, to the point that the simple entailment system morphisms are also, essentially, the morphisms of a Kleisli category for a monad in \mathbf{Ent} , the category of entailment systems and standard (*plain*) entailment system morphisms.

We point out that the definition of the composition of derivators, in strong contrast with that of polyderivators below, is based on the standard specification morphisms between Hall specifications. Actually, if instead of starting from a mapping $\varphi: S \longrightarrow T^*$, as is the case for the polyderivators, we start from an ordinary mapping $\varphi: S \longrightarrow T$, then, as we state next, we get a functor $(\varphi^* \times \varphi, h^\varphi)^*$ from the category $\mathbf{Alg}(\mathbf{H}_T)$, of Hall algebras for T , to the category $\mathbf{Alg}(\mathbf{H}_S)$, of Hall algebras for S (and the existence of such a functor will follow from that of a specification morphism from $(S^* \times S, \Sigma^{\mathbf{H}_S}, \mathcal{E}^{\mathbf{H}_S})$ to $(T^* \times T, \Sigma^{\mathbf{H}_T}, \mathcal{E}^{\mathbf{H}_T})$). This functor, in its turn, will allow us to endow with a structure of Hall algebra for S to the many-sorted set $\mathbf{HTer}_T(\Lambda)_{\varphi^* \times \varphi}$, from which the composition of derivators will be defined explicitly.

Proposition 47. *Let $\varphi: S \longrightarrow T$ be a mapping. Then the $S^* \times S$ -sorted mapping*

$$h^\varphi: \Sigma^{\mathbf{H}_S} \longrightarrow \Sigma_{\varphi^* \times \varphi}^{\mathbf{H}_T}$$

defined as follows

- (1) *For every $w \in S^*$ and $i \in |w|$, $h^\varphi(\pi_i^w) = \pi_i^{\varphi^*(w)}$,*
- (2) *For every $u, w \in S^*$ and $s \in S$, $h^\varphi(\xi_{u,w,s}) = \xi_{\varphi^*(u), \varphi^*(w), \varphi(s)}$,*

is such that $(\varphi^ \times \varphi, h^\varphi): (S^* \times S, \Sigma^{\mathbf{H}_S}, \mathcal{E}^{\mathbf{H}_S}) \longrightarrow (T^* \times T, \Sigma^{\mathbf{H}_T}, \mathcal{E}^{\mathbf{H}_T})$ is a specification morphism. Therefore $\varphi: S \longrightarrow T$ induces a functor $(\varphi^* \times \varphi, h^\varphi)^*$ from $\mathbf{Alg}(\mathbf{H}_T)$ to $\mathbf{Alg}(\mathbf{H}_S)$ which sends $\mathbf{HTer}_T(\Lambda)$, the free Hall algebra on a T -sorted signature Λ , to a Hall algebra for S , with $\mathbf{HTer}_T(\Lambda)_{\varphi^* \times \varphi}$ as underlying $S^* \times S$ -sorted set.*

For a derivator $\mathbf{d}: \Sigma \longrightarrow \Lambda$, the ms-mapping d from Σ to $\mathbf{HTer}_T(\Lambda)_{\varphi^* \times \varphi}$ can be lifted to a homomorphism of Hall algebras d^\sharp from $\mathbf{HTer}_S(\Sigma)$ to $\mathbf{HTer}_T(\Lambda)_{\varphi^* \times \varphi}$, whose underlying $S^* \times S$ -sorted mapping determines a translation of terms for Σ to terms for Λ . In particular, for every $(w, s) \in S^* \times S$, $d^\sharp_{w,s}$ assigns to terms in $\mathbf{T}_\Sigma(\downarrow w)_s$ terms in $\mathbf{T}_\Lambda(\downarrow \varphi^\sharp(w))_{\varphi(s)}$.

Before we define immediately below the composition of derivors we recall that Σ , Λ , Ω , and Ξ denote the signatures (S, Σ) , (T, Λ) , (U, Ω) , and (X, Ξ) , respectively, and \mathbf{d} , \mathbf{e} , and \mathbf{h} denote the derivors (φ, d) , (ψ, e) , and (γ, h) , respectively.

Definition 30. Let $\mathbf{d}: \Sigma \longrightarrow \Lambda$ and $\mathbf{e}: \Lambda \longrightarrow \Omega$ be derivors. Then $\mathbf{e} \circ \mathbf{d}$, the composition of \mathbf{d} and \mathbf{e} , is the derivor $(\psi \circ \varphi, e_{\varphi^* \times \varphi}^\# \circ d)$, where $e_{\varphi^* \times \varphi}^\# \circ d$ is obtained from

$$\begin{array}{ccc} \Lambda & \xrightarrow{\eta_\Lambda^{\text{HT}}} & \text{HTer}_T(\Lambda) \\ & \searrow e & \downarrow e^\# \\ & & \text{HTer}_U(\Omega)_{\psi^* \times \psi} \end{array} \quad \text{as} \quad \begin{array}{ccc} \text{HTer}_T(\Lambda)_{\varphi^* \times \varphi} & \xleftarrow{d} & \Sigma \\ \downarrow e_{\varphi^* \times \varphi}^\# & & \\ \text{HTer}_U(\Omega)_{\psi^* \times \psi, \varphi^* \times \varphi} & & \end{array}$$

where $e^\#$ is the extension of e to the free Hall algebra on Λ .

For every signature Σ , the identity at Σ is $(\text{id}_S, \eta_\Sigma^{\text{HS}})$.

The preceding definition allows us to get a corresponding, and explicit, category of signatures and derivors.

Proposition 48. *The signatures together with the derivors determine a category, that we denote by \mathbf{Sig}_δ .*

Proof. We restrict ourselves to prove that the composition of derivors is a derivor and that the composition is associative.

We have that

$$\begin{aligned} \text{HTer}_U(\Omega)_{\psi^* \times \psi, \varphi^* \times \varphi} &= \text{HTer}_U(\Omega)_{(\psi^* \times \psi) \circ (\varphi^* \times \varphi)} \\ &= \text{HTer}_U(\Omega)_{(\psi^* \circ \varphi^*) \times (\psi \circ \varphi)} \\ &= \text{HTer}_U(\Omega)_{(\psi \circ \varphi)^* \times (\psi \circ \varphi)}, \end{aligned}$$

hence the composition of derivors is a derivor.

Given the situation described by the following diagram

$$\Sigma \xrightarrow{\mathbf{d}} \Lambda \xrightarrow{\mathbf{e}} \Omega \xrightarrow{\mathbf{h}} \Xi,$$

we have that

$$\begin{aligned} \mathbf{h} \circ (\mathbf{e} \circ \mathbf{d}) &= \mathbf{h} \circ (\psi \circ \varphi, e_{\varphi^* \times \varphi}^\# \circ d) \\ &= (\gamma \circ (\psi \circ \varphi), h_{(\psi \circ \varphi)^* \times (\psi \circ \varphi)}^\# \circ (e_{\varphi^* \times \varphi}^\# \circ d)) \\ &= ((\gamma \circ \psi) \circ \varphi, h_{\psi^* \times \psi, \varphi^* \times \varphi}^\# \circ (e_{\varphi^* \times \varphi}^\# \circ d)) \\ &= ((\gamma \circ \psi) \circ \varphi, (h_{\psi^* \times \psi, \varphi^* \times \varphi}^\# \circ e_{\varphi^* \times \varphi}^\#) \circ d) \\ &= ((\gamma \circ \psi) \circ \varphi, (h_{\psi^* \times \psi}^\# \circ e^\#)_{\varphi^* \times \varphi} \circ d) \\ &= ((\gamma \circ \psi) \circ \varphi, (h_{\psi^* \times \psi}^\# \circ e)_{\varphi^* \times \varphi}^\# \circ d) \\ &= (\gamma \circ \psi, h_{\psi^* \times \psi}^\# \circ e) \circ \mathbf{d} \\ &= (\mathbf{h} \circ \mathbf{e}) \circ \mathbf{d}, \end{aligned}$$

therefore the composition of derivors is associative. \square

We point out that the category \mathbf{Sig}_δ of signatures and derivors can be obtained, naturally, as an isomorphic copy of the Kleisli category for a monad in \mathbf{Sig} . This is founded on the fact that, for every set of sorts S , we have the adjunction $\mathbf{T}_{\text{HS}} \dashv \mathbf{G}_{\text{HS}}$, from which we get the monad $\mathbf{T}_{\text{HS}} = (\mathbf{T}_{\text{HS}}, \eta^{\text{HS}}, \mu^{\text{HS}})$ in $\mathbf{Set}^{S^* \times S}$, that canonically induces the monad in \mathbf{Sig} at issue.

Proposition 49. *The triple $\mathbb{T}_\mathfrak{d} = (\mathfrak{d}, \eta^\mathfrak{d}, \mu^\mathfrak{d})$, where*

- (1) \mathfrak{d} is the functor which sends a signature Σ to the signature $(S, \mathbb{T}_{\mathbb{H}_S}(\Sigma))$, and a signature morphism \mathbf{d} from Σ to Λ to the signature morphism $(\varphi, d^\#)$ from $(S, \mathbb{T}_{\mathbb{H}_S}(\Sigma))$ to $(T, \mathbb{T}_{\mathbb{H}_T}(\Lambda))$,
- (2) $\eta_\Sigma^\mathfrak{d} = (\text{id}_S, \eta_\Sigma^{\mathbb{H}_S})$, with $\eta_\Sigma^{\mathbb{H}_S}$ the value at Σ of the unit $\eta^{\mathbb{H}_S}$ of the monad $\mathbb{T}_{\mathbb{H}_S}$, and
- (3) $\mu_\Sigma^\mathfrak{d} = (\text{id}_S, \mu_\Sigma^{\mathbb{H}_S})$, with $\mu_\Sigma^{\mathbb{H}_S}$ the value at Σ of the multiplication $\mu^{\mathbb{H}_S}$ of the monad $\mathbb{T}_{\mathbb{H}_S}$,

is a monad in \mathbf{Sig} and the categories $\mathbf{Sig}_\mathfrak{d}$ and $\mathbf{Kl}(\mathbb{T}_\mathfrak{d})$ are isomorphic.

Remark. Almost all the results about the categories \mathbf{Sig} , \mathbf{Alg} and \mathbf{Spf} established in the second and third section, suitably extended, are also valid for the corresponding categories $\mathbf{Sig}_\mathfrak{d}$, $\mathbf{Alg}_\mathfrak{d}$ and $\mathbf{Spf}_\mathfrak{d}$. But being the derivors a particular case of the polyderivors, we restrict ourselves to unfold those results only for the polyderivors.

We state now a lemma from which the existence of coproducts in $\mathbf{Sig}_\mathfrak{d}$ follows immediately.

Lemma 7. *Let \mathbb{T} be a monad in a category \mathbf{C} . If \mathbf{C} has coproducts, then $\mathbf{Kl}(\mathbb{T})$ has coproducts.*

Proof. Let $(X_i)_{i \in I}$ be a family of objects in $\mathbf{Kl}(\mathbb{T})$. Then $\coprod_{i \in I} X_i$, together with the family of morphisms $(\eta_{\coprod_{i \in I} X_i} \circ \text{in}_i)_{i \in I}$ where, for $i \in I$, in_i is the structural morphism from X_i into $\coprod_{i \in I} X_i$, is a coproduct in $\mathbf{Kl}(\mathbb{T})$ of $(X_i)_{i \in I}$.

Let $(f_i: X_i \rightarrow Y)_{i \in I}$ be a family of morphisms in $\mathbf{Kl}(\mathbb{T})$. Then, by the universal property of the coproduct, there exists a unique morphism $[f_i]_{i \in I}$, in \mathbf{C} , from $\coprod_{i \in I} X_i$ into $T(Y)$ such that, for every $i \in I$, $[f_i]_{i \in I} \circ \text{in}_i = f_i$. Furthermore, we have that

$$\begin{aligned} [f_i]_{i \in I} &= [f_i]_{i \in I} \diamond \eta_{\coprod_{i \in I} X_i} && \text{(since } \eta_{\coprod_{i \in I} X_i} \text{ is neutral for } \diamond) \\ &= \mu_Y \circ T([f_i]_{i \in I}) \circ \eta_{\coprod_{i \in I} X_i} && \text{(by definition of } \diamond). \end{aligned}$$

Therefore we get the commutative diagram

$$\begin{array}{ccccc} X_i & \xrightarrow{\text{in}_i} & \coprod_{i \in I} X_i & \xrightarrow{\eta_{\coprod_{i \in I} X_i}} & T(\coprod_{i \in I} X_i) \\ & \searrow f_i & \downarrow [f_i]_{i \in I} & & \downarrow T([f_i]_{i \in I}) \\ & & T(Y) & \xleftarrow{\mu_Y} & T(T(Y)) \end{array}$$

and we can assert that, for every $i \in I$, $[f_i]_{i \in I} \diamond (\eta_{\coprod_{i \in I} X_i} \circ \text{in}_i) = f_i$. To prove the uniqueness, let g be a morphism, in $\mathbf{Kl}(\mathbb{T})$, from $\coprod_{i \in I} X_i$ into Y such that, for every $i \in I$, $g \diamond (\eta_{\coprod_{i \in I} X_i} \circ \text{in}_i) = f_i$. Then, for $i \in I$, we have that

$$\begin{aligned} f_i &= g \diamond (\eta_{\coprod_{i \in I} X_i} \circ \text{in}_i) \\ &= \mu_Y \circ T(g) \circ (\eta_{\coprod_{i \in I} X_i} \circ \text{in}_i) && \text{(by definition of } \diamond) \\ &= (\mu_Y \circ T(g) \circ \eta_{\coprod_{i \in I} X_i}) \circ \text{in}_i \\ &= (g \diamond \eta_{\coprod_{i \in I} X_i}) \circ \text{in}_i && \text{(by definition of } \diamond) \\ &= g \circ \text{in}_i && \text{(since } \eta_{\coprod_{i \in I} X_i} \text{ is neutral for } \diamond), \end{aligned}$$

thus $g = [f_i]_{i \in I}$. Therefore $[f_i]_{i \in I}: \coprod_{i \in I} X_i \rightarrow Y$ in $\mathbf{Kl}(\mathbb{T})$ satisfies the universal property. \square

Corollary 14. *The category $\mathbf{Sig}_\mathfrak{d}$ has coproducts.*

Proof. Because **Sig** has coproducts, the category $\mathbf{Kl}(\mathbb{T}_\delta)$, by Lemma 7, has also coproducts. Therefore, since \mathbf{Sig}_δ and $\mathbf{Kl}(\mathbb{T}_\delta)$ are isomorphic, the category \mathbf{Sig}_δ has coproducts. \square

After having stated the above facts about the derivors, we are in the position to introduce the definition of the composition of two polyderivors, in order to get the corresponding category \mathbf{Sig}_{p_δ} of signatures and polyderivors. To this end we begin by stating that every mapping $\varphi: S \longrightarrow T^*$ gives rise to a functor $(\varphi^\sharp \times \varphi^\sharp, b^\varphi)^*$ from $\mathbf{Alg}(B_T)$, the category of Bénabou algebras for T , to $\mathbf{Alg}(B_S)$, the category of Bénabou algebras for S (observe that such a functor is induced not by a standard specification morphism between Bénabou specifications, but by a *derivor* b^φ between the corresponding Bénabou signatures). This functor, in its turn, will allow us to endow the many-sorted set $\mathbf{BTer}_T(\Lambda)_{\varphi^\sharp \times \varphi^\sharp}$ with a structure of Bénabou algebra for S , from which the definition of the composition of polyderivors will follow.

Proposition 50. *Let φ be a mapping from S to T^* . Then the $((S^*)^2)^* \times (S^*)^2$ -sorted mapping*

$$b^\varphi: \Sigma^{\mathbf{B}S} \longrightarrow \mathbf{HTer}_{T^* \times T^*}(\Sigma^{\mathbf{B}T})_{(\varphi^\sharp \times \varphi^\sharp)^* \times (\varphi^\sharp \times \varphi^\sharp)}$$

defined as follows

- (1) For every $w \in S^*$ and $\alpha \in |w|$, $b^\varphi(\pi_\alpha^w)$ is the $\Sigma^{\mathbf{B}T}$ -term

$$\langle \pi_{\sum_{\beta \in \alpha} p_\beta}^{\varphi^\sharp(w)}, \dots, \pi_{\sum_{\beta \in \alpha+1} p_{\beta-1}}^{\varphi^\sharp(w)} \rangle_{\varphi^\sharp(w), \varphi(w_\alpha)}$$

of type $\lambda \longrightarrow (\varphi^\sharp(w), (\varphi(w_\alpha)))$,

- (2) For every $u, w \in S^*$, $b^\varphi(\langle \rangle_{u,w})$ is the $\Sigma^{\mathbf{B}T}$ -term

$$\begin{aligned} & \langle \pi_0^{\varphi(w_0)} \circ v_0^{(\varphi^\sharp(u), \varphi(w_0))}, \dots, \pi_{|\varphi(w_0)|-1}^{\varphi(w_0)} \circ v_0^{(\varphi^\sharp(u), \varphi(w_0))}, \dots, \\ & \pi_0^{\varphi(w_i)} \circ v_i^{(\varphi^\sharp(u), \varphi(w_i))}, \dots, \pi_{|\varphi(w_i)|-1}^{\varphi(w_i)} \circ v_i^{(\varphi^\sharp(u), \varphi(w_i))}, \dots, \\ & \pi_0^{\varphi(w_{|w|-1})} \circ v_{|w|-1}^{(\varphi^\sharp(u), \varphi(w_{|w|-1}))}, \dots, \pi_{|\varphi(w_{|w|-1})|-1}^{\varphi(w_{|w|-1})} \circ v_{|w|-1}^{(\varphi^\sharp(u), \varphi(w_{|w|-1}))} \rangle \end{aligned}$$

of type $((\varphi^\sharp(u), \varphi(w_0)), \dots, (\varphi^\sharp(u), \varphi(w_{|w|-1}))) \longrightarrow (\varphi^\sharp(u), \varphi^\sharp(w))$,

- (3) For every $u, x, w \in S^*$, $b^\varphi(\circ_{u,x,w})$ is the $\Sigma^{\mathbf{B}T}$ -term

$$\circ_{\varphi^\sharp(u), \varphi^\sharp(x), \varphi^\sharp(w)}(v_0^{(\varphi^\sharp(u), \varphi^\sharp(x))}, v_1^{(\varphi^\sharp(x), \varphi^\sharp(w))})$$

of type $((\varphi^\sharp(u), \varphi^\sharp(x)), (\varphi^\sharp(x), \varphi^\sharp(w))) \longrightarrow (\varphi^\sharp(u), \varphi^\sharp(w))$,

is such that $(\varphi^\sharp \times \varphi^\sharp, b^\varphi): (S^* \times S^*, \Sigma^{\mathbf{B}S}, \mathcal{E}^{\mathbf{B}S}) \longrightarrow (T^* \times T^*, \Sigma^{\mathbf{B}T}, \mathcal{E}^{\mathbf{B}T})$ is a specification morphism. Therefore $\varphi: S \longrightarrow T^*$ induces a functor $(\varphi^\sharp \times \varphi^\sharp, b^\varphi)^*$ from $\mathbf{Alg}(B_T)$ to $\mathbf{Alg}(B_S)$ which sends $\mathbf{BTer}_T(\Lambda)$, the free Bénabou algebra on the T -sorted signature Λ , to a Bénabou algebra for S , with $\mathbf{BTer}_T(\Lambda)_{\varphi^\sharp \times \varphi^\sharp}$ as underlying $S^* \times S^*$ -sorted set.

For a polyderivor $\mathbf{d}: \Sigma \longrightarrow \Lambda$, we can extend the $S^* \times S^*$ -sorted mapping d from $\prod_{1 \times \check{\Sigma}} \Sigma$ to $\mathbf{BTer}_T(\Lambda)_{\varphi^\sharp \times \varphi^\sharp}$ to a homomorphism of Bénabou algebras d^\sharp from $\mathbf{BTer}_S(\Sigma)$ to $\mathbf{BTer}_T(\Lambda)_{\varphi^\sharp \times \varphi^\sharp}$, whose underlying $S^* \times S^*$ -mapping determines a translation of terms for Σ into terms for Λ . In particular, for every $(w, s) \in S^* \times S$, $d_{w,(s)}^\sharp$ assigns to terms in $\mathbf{T}_\Sigma(\downarrow w)_s$ terms in $\mathbf{T}_\Lambda(\downarrow \varphi^\sharp(w))_{\varphi(s)}$, in such a way that to a variable v_α^s in $\downarrow w$ associates the family of variables

$$(v_{\sum_{\beta \in \alpha} p_\beta}^{\varphi^\sharp(w)}, \dots, v_{\sum_{\beta \in \alpha+1} p_{\beta-1}}^{\varphi^\sharp(w)}).$$

Before we define next the composition of polyderivors we recall that Σ , Λ , Ω , and Ξ denote the signatures (S, Σ) , (T, Λ) , (U, Ω) , and (X, Ξ) , respectively, and \mathbf{d} , \mathbf{e} , and \mathbf{h} denote the polyderivors (φ, d) , (ψ, e) , and (γ, h) , respectively.

Definition 31. Let $\mathbf{d}: \Sigma \rightarrow \Lambda$ and $\mathbf{e}: \Lambda \rightarrow \Omega$ be polyderivors. Then the *composition* of \mathbf{d} and \mathbf{e} , denoted by $\mathbf{e} \circ \mathbf{d}$, is the morphism $(\psi^\# \circ \varphi, e_{\varphi^\# \times \psi^\#}^\# \circ d)$, where the first component $\psi^\# \circ \varphi$ is a mapping from S to U^* and $e_{\varphi^\# \times \psi^\#}^\# \circ d$ is obtained from

$$\begin{array}{ccc} \coprod_{1 \times \check{\Omega}_T} \Lambda & \xrightarrow{\eta_{\coprod_{1 \times \check{\Omega}_T} \Lambda}^{\text{BT}}} & \text{BTer}_T(\Lambda) & & \text{BTer}_T(\Lambda)_{\varphi^\# \times \psi^\#} & \xleftarrow{d} & \coprod_{1 \times \check{\Omega}_S} \Sigma \\ & \searrow e & \downarrow e^\# & \text{as} & \downarrow e_{\varphi^\# \times \psi^\#}^\# & & \\ & & \text{BTer}_U(\Omega)_{\psi^\# \times \psi^\#} & & \text{BTer}_U(\Omega)_{\psi^\# \times \psi^\# \varphi^\# \times \psi^\#} & & \end{array}$$

and, for every signature Σ , the *identity* at Σ is the polydivor $(\check{\Omega}_S, \eta_\Sigma^{\text{BS}})$.

From this definition we get the corresponding category of signatures and polyderivors.

Proposition 51. *The signatures together with the polyderivors determine a category, that we denote by \mathbf{Sig}_{pd} .*

Proof. To begin with we prove that the composition of polyderivors is a polydivor.

$$\begin{aligned} (\text{BTer}_U(\Omega)_{\psi^\# \times \psi^\#})_{\varphi^\# \times \psi^\#} &= \text{BTer}_U(\Omega)_{(\psi^\# \times \psi^\#) \circ (\varphi^\# \times \psi^\#)} \\ &= \text{BTer}_U(\Omega)_{(\psi^\# \circ \varphi^\#) \times (\psi^\# \circ \psi^\#)} \\ &= \text{BTer}_U(\Omega)_{(\psi^\# \circ \varphi)^\# \times (\psi^\# \circ \psi)^\#}. \end{aligned}$$

Next we prove that the identities are true identities.

$$\begin{aligned} \mathbf{d} \circ (\check{\Omega}_S, \eta_\Sigma^{\text{BS}}) &= (\varphi^\# \circ \check{\Omega}_S, d_{\check{\Omega}_S^\# \times \check{\Omega}_S^\#}^\# \circ \eta_\Sigma^{\text{BS}}) \\ &= \mathbf{d}, \end{aligned}$$

$$\begin{aligned} (\check{\Omega}_T, \eta_\Lambda^{\text{BT}}) \circ \mathbf{d} &= (\check{\Omega}_T^\# \circ \varphi, (\eta_\Lambda^{\text{BT}})_{\varphi^\# \times \psi^\#}^\# \circ d) \\ &= \mathbf{d}. \end{aligned}$$

Finally we prove that the composition is associative. Given the morphisms

$$\Sigma \xrightarrow{\mathbf{d}} \Lambda \xrightarrow{\mathbf{e}} \Omega \xrightarrow{\mathbf{h}} \Xi,$$

we have that

$$\begin{aligned} \mathbf{h} \circ (\mathbf{e} \circ \mathbf{d}) &= \mathbf{h} \circ (\psi^\# \circ \varphi, e_{\varphi^\# \times \psi^\#}^\# \circ d) \\ &= (\gamma^\# \circ (\psi^\# \circ \varphi), h_{(\psi^\# \circ \varphi)^\# \times (\psi^\# \circ \psi)^\#}^\# \circ (e_{\varphi^\# \times \psi^\#}^\# \circ d)) \\ &= ((\gamma^\# \circ \psi)^\# \circ \varphi, h_{\psi^\# \times \psi^\# \varphi^\# \times \psi^\#}^\# \circ (e_{\varphi^\# \times \psi^\#}^\# \circ d)) \\ &= ((\gamma^\# \circ \psi)^\# \circ \varphi, (h_{\psi^\# \times \psi^\# \varphi^\# \times \psi^\#}^\# \circ e_{\varphi^\# \times \psi^\#}^\#) \circ d) \\ &= ((\gamma^\# \circ \psi)^\# \circ \varphi, (h_{\psi^\# \times \psi^\# \varphi^\# \times \psi^\#}^\# \circ e^\#)_{\varphi^\# \times \psi^\#} \circ d) \\ &= ((\gamma^\# \circ \psi)^\# \circ \varphi, (h_{\psi^\# \times \psi^\# \varphi^\# \times \psi^\#}^\# \circ e)_{\varphi^\# \times \psi^\#}^\# \circ d) \\ &= (\gamma^\# \circ \psi, h_{\psi^\# \times \psi^\#}^\# \circ e) \circ \mathbf{d} \\ &= (\mathbf{h} \circ \mathbf{e}) \circ \mathbf{d}. \end{aligned}$$

□

Remark. From the fact that $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ is a category it follows at once that, for every signature Σ in $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$, the set of all endopolyderivors of Σ , $\mathbf{End}_{\mathfrak{p}\mathfrak{d}}(\Sigma)$, is the underlying set of a monoid, denoted by $\mathbf{End}_{\mathfrak{p}\mathfrak{d}}(\Sigma)$. Since the monoid $\mathbf{End}_{\mathfrak{d}}(\Sigma)$, of hypersubstitutions of Σ , i.e, the monoid of endoderivors of Σ , is embedded (in general, strictly) in the monoid $\mathbf{End}_{\mathfrak{p}\mathfrak{d}}(\Sigma)$, we conclude that $\mathbf{End}_{\mathfrak{p}\mathfrak{d}}(\Sigma)$ can serve as a basis to develop a doubly generalized (because of the use of many-sorted signatures and endopolyderivors, instead of single-sorted signatures and endoderivors,) theory of hyperidentities. But we leave this task for another occasion.

Having shown above that the concept of derivor, because of its reducibility to that of morphism of a Kleisli category for a monad in \mathbf{Sig} , is mathematically natural, one could also expect to show the mathematical naturalness of the notion of polyderivor by proving that the category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ is obtainable as an isomorphic copy of the Kleisli category for some monad in \mathbf{Sig} . This is actually true, however the procedure we should follow to determine such a monad is more involved than the one, relatively simple, we have followed for the derivors. This is due to the fact that, for a signature $\Sigma = (S, \Sigma)$, the pair $(S^* \times S^*, \mathbf{B}\mathbf{T}\mathbf{er}_S(\Sigma))$ is not a signature, because $\mathbf{B}\mathbf{T}\mathbf{er}_S(\Sigma)$ is an $S^* \times S^*$ -sorted set, but not an $((S^*)^2)^* \times (S^*)^2$ -sorted set.

The approach we offer to prove the existence of the monad in \mathbf{Sig} whose Kleisli category is isomorphic to $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ will be based, on the one hand, on the functor

$$\Delta_{\lambda_S \times 1}: \mathbf{Set}^{S^* \times S^*} \longrightarrow \mathbf{Set}^{S^{**} \times S^*}$$

which sends $S^* \times S^*$ -sorted sets to S^* -signatures, therefore, for an S -sorted signature Σ , we will have that $\Delta_{\lambda_S \times 1}(\mathbf{B}\mathbf{T}\mathbf{er}_S(\Sigma))$ is a S^* -signature, and, on the other hand, of the fact that, for every set of sorts S , the adjunction $\mathbf{T}_{B_S} \dashv \mathbf{G}_{B_S}$, determines a monad on $\mathbf{Set}^{S^* \times S^*}$ denoted as $\mathbb{T}_{B_S} = (\mathbf{T}_{B_S}, \eta^{B_S}, \mu^{B_S})$.

Proposition 52. *There exists a monad $\mathbb{T}_{\mathfrak{p}\mathfrak{d}} = (\mathfrak{p}\mathfrak{d}, \eta^{\mathfrak{p}\mathfrak{d}}, \mu^{\mathfrak{p}\mathfrak{d}})$ in \mathbf{Sig} such that the categories $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ and $\mathbf{Kl}(\mathbb{T}_{\mathfrak{p}\mathfrak{d}})$ are isomorphic.*

Proof. Let $\mathfrak{p}\mathfrak{d}$ be the endofunctor of \mathbf{Sig} defined as follows

- (1) $\mathfrak{p}\mathfrak{d}$ sends a signature Σ to

$$(S^*, \mathbf{T}_{B_S}(\coprod_{1 \times \check{Q}_S} \Sigma)_{\lambda_S \times 1}).$$

- (2) $\mathfrak{p}\mathfrak{d}$ sends a signature morphism \mathbf{d} from Σ to Λ to

$$(\varphi^*, (d^\#)_{\lambda_S \times 1}): (S^*, \mathbf{T}_{B_S}(\coprod_{1 \times \check{Q}_S} \Sigma)_{\lambda_S \times 1}) \longrightarrow (T^*, \mathbf{T}_{B_T}(\coprod_{1 \times \check{Q}_T} \Lambda)_{\lambda_T \times 1}),$$

where $\mathbf{T}_{B_S}(\coprod_{1 \times \check{Q}_S} \Sigma)_{\lambda_S \times 1}$ is the value in Σ of the functor

$$\mathbf{Set}^{S^* \times S} \xrightarrow{\coprod_{1 \times \check{Q}_S}} \mathbf{Set}^{S^* \times S^*} \xrightarrow{\mathbf{T}_{B_S}} \mathbf{Set}^{S^* \times S^*} \xrightarrow{\Delta_{\lambda_S \times 1}} \mathbf{Set}^{S^{**} \times S^*}.$$

After having defined the endofunctor $\mathfrak{p}\mathfrak{d}$ of \mathbf{Sig} , we proceed to define the unit $\eta^{\mathfrak{p}\mathfrak{d}}$ and multiplication $\mu^{\mathfrak{p}\mathfrak{d}}$ of the monad $\mathbb{T}_{\mathfrak{p}\mathfrak{d}}$.

Let Σ be a signature. Then we have that $\eta_{\Sigma}^{\mathfrak{p}\mathfrak{d}}$, the component of the unit $\eta^{\mathfrak{p}\mathfrak{d}}$ of the purported monad $\mathbb{T}_{\mathfrak{p}\mathfrak{d}}$ in Σ , is the signature morphism $(\check{Q}_S, \eta_{\Sigma}^{B_S})$, i.e., the value in Σ of the unit of the monad $\mathbb{T}_{B_S} = (\mathbf{T}_{B_S}, \eta^{B_S}, \mu^{B_S})$ in $\mathbf{Set}^{S^* \times S^*}$, obtained from the adjunction $\mathbf{T}_{B_S} \dashv \mathbf{G}_{B_S}$. On the other hand, we want $\mu_{\Sigma}^{\mathfrak{p}\mathfrak{d}}$, the component of the multiplication $\mu^{\mathfrak{p}\mathfrak{d}}$ of the purported monad $\mathbb{T}_{\mathfrak{p}\mathfrak{d}}$ in Σ , to be a morphism as in the

following diagram

$$\begin{array}{c} (S^{**}, T_{B_{S^*}}(\coprod_{1 \times \check{Q}_{S^*}}(T_{B_S}(\coprod_{1 \times \check{Q}_S} \Sigma)_{\lambda_S \times 1}))_{\lambda_{S^*} \times 1}) \\ \downarrow \mu_{\Sigma}^{\text{pd}} \\ (S^*, T_{B_S}(\coprod_{1 \times \check{Q}_S} \Sigma)_{\lambda_S \times 1}) \end{array}$$

The first coordinate of μ_{Σ}^{pd} is λ_S , the multiplication of the monad T_* . To get the second coordinate of μ_{Σ}^{pd} we have to define a natural transformation α as in the following diagram

$$\begin{array}{ccccc} & & \text{Set}^{S^{**} \times S^{**}} & \xrightarrow{T_{B_{S^*}}} & \text{Set}^{S^{**} \times S^{**}} & \xrightarrow{\Delta_{\lambda_{S^*} \times 1}} & \text{Set}^{S^{**} \times S^{**}} \\ & & \uparrow & & \uparrow & & \uparrow \\ \coprod_{1 \times \check{Q}_{S^*}} & & \text{Set}^{S^{**} \times S^*} & \xrightarrow{\alpha} & \Delta_{\lambda_S \times \lambda_S} & = & \Delta_{\lambda_{S^*} \times \lambda_S} \\ & & \uparrow & & \uparrow & & \uparrow \\ \text{Set}^{S^* \times S^*} & \xrightarrow{T_{B_S}} & \text{Set}^{S^* \times S^*} & \xrightarrow{T_{B_S}} & \text{Set}^{S^* \times S^*} & \xrightarrow{\Delta_{\lambda_S \times 1}} & \text{Set}^{S^{**} \times S^*} \\ & & \downarrow \mu^{B_S} & & \uparrow & & \\ \coprod_{1 \times \check{Q}_S} & \uparrow & \text{Set}^{S^* \times S} & \xrightarrow{T_{B_S}} & \text{Set}^{S^* \times S} & & \\ & & \text{Set}^{S^* \times S} & & & & \end{array}$$

Let Θ be an $S^* \times S^*$ -sorted set. Then $T_{B_S}(\Theta)_{\lambda_S \times \lambda_S}$ has a natural structure of $\Sigma^{B_{S^*}}$ -algebra, obtained from the $(S^{**} \times S^{**})^* \times (S^{**} \times S^{**})$ -sorted mapping $b^{\lambda_S}: \Sigma^{B_{S^*}} \longrightarrow \text{Ter}_{S^* \times S^*}(\Sigma^{B_S})_{(\lambda_S \times \lambda_S)^* \times (\lambda_S \times \lambda_S)}$ by applying Proposition 50 to the mapping $\lambda_S: S^{**} \longrightarrow S^*$.

On the other hand, for every $S^* \times S^*$ -sorted set Θ , we have an $S^{**} \times S^{**}$ -sorted mapping f_{Θ} from $\coprod_{1 \times \check{Q}_{S^*}}(\Delta_{\lambda_S \times 1}(\Theta))$ to $\Delta_{\lambda_S \times \lambda_S}(T_{B_S}(\Theta))$ which, for every $(\bar{u}, \bar{w}) \in S^{**} \times S^{**}$, assigns to an element P , the image of P under the inclusion $\eta_{\Theta}^{B_S}$ of Θ into $T_{B_S}(\Theta)$. The definition is sound because, in this case, \bar{w} has the form (w) , P is in $\Theta_{\lambda_S u, w}$ and $\eta_{\Theta}^{B_S}(P)$ belongs to $\Delta_{\lambda_S \times \lambda_S}(T_{B_S}(\Theta))$. Then the extension f_{Θ}^{\sharp} of f_{Θ} to $T_{B_{S^*}}(\coprod_{1 \times \check{Q}_{S^*}}(\Delta_{\lambda_S \times 1}(\Theta)))$ is the component at Θ of the natural transformation α .

Therefore, the second coordinate of μ_{Σ}^{pd} is the value at Σ of the natural transformation

$$(\Delta_{\lambda_{S^*} \times \lambda_S} * \Delta_{\lambda_S \times 1} * \mu^{B_S} * \coprod_{1 \times \check{Q}_S}) \circ (\Delta_{\lambda_{S^*} \times 1} * \alpha * T_{B_S} * \coprod_{1 \times \check{Q}_{S^*}}).$$

Finally we prove that the categories \mathbf{Sig}_{pd} and $\mathbf{Kl}(T_{\text{pd}})$ are isomorphic.

A morphism $\mathbf{d}: \Sigma \longrightarrow \mathbf{A}$ in $\mathbf{Kl}(T_{\text{pd}})$ is a morphism $\mathbf{d}: \Sigma \longrightarrow \text{pd}(\mathbf{A})$ in \mathbf{Sig} , hence $\varphi: S \longrightarrow T^*$ and

$$\begin{aligned} d: \Sigma &\longrightarrow \Delta_{\varphi^* \times \varphi}(T_{B_T}(\coprod_{1 \times \check{Q}_T} \mathbf{A})_{\lambda_T \times 1}) \\ &= \Delta_{\varphi^{\sharp} \times \varphi}(T_{B_T}(\coprod_{1 \times \check{Q}_T} \mathbf{A})) \\ &\cong \Delta_{\varphi^{\sharp} \times \varphi}(\text{BTer}_T(\mathbf{A})), \end{aligned}$$

that is exactly the definition of polyderivor in \mathbf{Sig}_{pd} . \square

Remark. From the existence of the monad T_{pd} in \mathbf{Sig} it follows the existence of an adjunction $F_{\text{pd}} \dashv G_{\text{pd}}$ from \mathbf{Sig} to $\mathbf{Kl}(T_{\text{pd}})$ which, in its turn, defines in \mathbf{Sig} exactly

the monad \mathbb{T}_{pd} (recall that the functor F_{pd} from \mathbf{Sig} to $\mathbf{Kl}(\mathbb{T}_{\text{pd}})$ sends a signature morphism \mathbf{d} from Σ to Λ in \mathbf{Sig} to the composite signature morphism

$$\Sigma \xrightarrow{\mathbf{d}} \Lambda \xrightarrow{\eta_{\Lambda}^{\text{pd}}} \text{pd}(\Lambda)$$

in $\mathbf{Kl}(\mathbb{T}_{\text{pd}})$; and that the functor G_{pd} from $\mathbf{Kl}(\mathbb{T}_{\text{pd}})$ to \mathbf{Sig} sends a morphism \mathbf{d} from Σ to Λ in $\mathbf{Kl}(\mathbb{T}_{\text{pd}})$ to the composite signature morphism

$$\text{pd}(\Sigma) \xrightarrow{\text{pd}(\mathbf{d})} \text{pd}(\text{pd}(\Lambda)) \xrightarrow{\mu_{\Lambda}^{\text{pd}}} \text{pd}(\Lambda)$$

in \mathbf{Sig}). Therefore, from the functor F_{pd} and taking into account the isomorphism between $\mathbf{Kl}(\mathbb{T}_{\text{pd}})$ and \mathbf{Sig}_{pd} , we get, automatically, a result stated in a laborious way in a previous example: that the concept of standard signature morphism is a particular case of that of polyderivator.

Corollary 15. *The category \mathbf{Sig}_{pd} has coproducts.*

Proof. Because \mathbf{Sig} has coproducts, the category $\mathbf{Kl}(\mathbb{T}_{\text{pd}})$, by Lemma 7, has also coproducts. Therefore, since \mathbf{Sig}_{pd} and $\mathbf{Kl}(\mathbb{T}_{\text{pd}})$ are isomorphic, the category \mathbf{Sig}_{pd} has coproducts. \square

After stating that a polyderivator, as was the case for a derivator, is nothing more (nor less) than a morphism of a Kleisli category for a convenient monad in \mathbf{Sig} , therefore confirming category-theoretically its naturalness, our next goal is to lift the contravariant functor $\text{Alg}: \mathbf{Sig} \rightarrow \mathbf{Cat}$ up to a contravariant pseudo-functor $\text{Alg}_{\text{pd}}: \mathbf{Sig}_{\text{pd}} \rightarrow \mathbf{Cat}$, that will allow us, by applying, once more, the construction of Ehresmann-Grothendieck, to get a new category of algebras \mathbf{Alg}_{pd} into which is embedded the category \mathbf{Alg} . But to achieve the just stated objective we should define beforehand some auxiliary functors and natural transformations.

Proposition 53. *Let S be a set of sorts. Then we have that*

- (1) *There exists an expansion functor $(\cdot)^{\natural_S}$ from \mathbf{Set}^S to \mathbf{Set}^{S^*} which sends an S -sorted set $A = (A_s)_{s \in S}$ to the S^* -sorted set $A^{\natural_S} = (A_w)_{w \in S^*}$, and an S -sorted mapping f from A to B to the S^* -sorted mapping $f^{\natural_S} = (f_w)_{w \in S^*}$ from $(A_w)_{w \in S^*}$ to $(B_w)_{w \in S^*}$. If A is an S -sorted set and $f: A \rightarrow B$ an S -sorted mapping, then we say that A^{\natural_S} and f^{\natural_S} are the expansions of A and f , respectively, to the words on S and, to simplify the notation, we will write A^{\natural} and f^{\natural} instead of A^{\natural_S} and f^{\natural_S} , respectively.*
- (2) *From the contravariant functor MSet , from \mathbf{Set} to \mathbf{Cat} , to the contravariant functor $\text{MSet} \circ \mathbf{T}_{\star}^{\text{op}}$ between the same categories, where $\mathbf{T}_{\star}^{\text{op}}$ is the composite of $\mathbf{T}_{\star}^{\text{op}}$ (the dual of the free monoid functor \mathbf{T}_{\star} from \mathbf{Set} to \mathbf{Mon} , the category of monoids), and $G_{\mathbf{Mon}}$ (the forgetful functor from \mathbf{Mon} to \mathbf{Set}), there exists a natural transformation $(\cdot)^{\natural}$ which sends a set S to the expansion functor $(\cdot)^{\natural_S}$ from \mathbf{Set}^S to \mathbf{Set}^{S^*} .*
- (3) *There exists a natural isomorphism ι_S from the functor $(\cdot)^{\natural_{S^*}} \circ (\cdot)^{\natural_S}$ to the functor $\Delta_{\wedge_S} \circ (\cdot)^{\natural_S}$, both from the category \mathbf{Set}^S to the category $\mathbf{Set}^{S^{**}}$.*

Proof. We restrict ourselves to prove the second and third parts of the proposition.

(2) $(\cdot)^\natural$ is a natural transformation from \mathbf{MSet} to $\mathbf{MSet} \circ \mathbf{T}_*^{\text{op}}$ since, for a mapping $\varphi: S \longrightarrow T$, the following diagram commutes

$$\begin{array}{ccc} \mathbf{Set}^S & \xrightarrow{(\cdot)^\natural_S} & \mathbf{Set}^{S^*} \\ \Delta_\varphi \uparrow & & \uparrow \Delta_{\varphi^*} \\ \mathbf{Set}^T & \xrightarrow{(\cdot)^\natural_T} & \mathbf{Set}^{T^*} \end{array}$$

Observe, in particular, that for a T -sorted set B , we have that $(B_\varphi)^\natural_S = (B^\natural_T)_{\varphi^*}$.

(3) It is enough to define, for every S -sorted set A , the component $(\iota_S)_A$ of ι_S at A , as the S^{**} -isomorphism $(\iota_S)_A: A^\natural \longrightarrow (A^\natural)_\lambda$ that has as \bar{w} -th coordinate, for $\bar{w} = (w_\alpha)_{\alpha \in |\bar{w}|} \in S^{**}$, the canonical isomorphism

$$A_{\bar{w}}^\natural = \prod_{\alpha \in |\bar{w}|} \prod_{j \in |w_\alpha|} A_{w_{\alpha j}} \xrightarrow{\langle \text{pr}_{\alpha_j} \circ \text{pr}_\alpha \rangle_{\alpha \in |\bar{w}|, j \in |w_\alpha|}} \prod_{\alpha \in |\bar{w}|, j \in |w_\alpha|} A_{w_{\alpha j}} = A_{\lambda \bar{w}}^\natural,$$

where $\text{pr}_\alpha: A_{\bar{w}} \longrightarrow A_{w_\alpha}$ and $\text{pr}_{\alpha_j}: A_{w_\alpha} \longrightarrow A_{w_{\alpha j}}$ are the canonical projections. To simplify the notation we will write ι^A instead of $(\iota_S)_A$. \square

Corollary 16. *Let $\varphi: S \longrightarrow T^*$ and $\psi: T \longrightarrow U^*$ be mappings. Then, for every T -sorted set B and U -sorted set C , we have that*

- (1) $((B^\natural_T)^\natural_{T^*})_{\varphi^*}$, denoted by B_{φ^*} , and $(B^\natural_T)_{\varphi^\sharp}$, denoted by B_{φ^\sharp} , are isomorphic S^* -sorted sets.
- (2) $((C^\natural_U)^\natural_\psi)_{\varphi}$, denoted by C_{ψ_φ} , and $(C^\natural_U)_{\psi^\sharp \circ \varphi}$, denoted by $C_{\psi^\sharp \circ \varphi}$, are isomorphic S -sorted sets.
- (3) There exists an isomorphism $\kappa_\varphi^B: \text{BOP}_T(B)_{\varphi^\sharp \times \varphi^\sharp} \longrightarrow \text{BOP}_S(B_\varphi)$, where, to simplify, we have written B_φ instead of $(B^\natural_T)_\varphi$.

Proof. (1) The isomorphism is $\iota_{\varphi^*}^B = (\iota_{\varphi^*(w)}^B: B_{\varphi^*(w)} \longrightarrow B_{\varphi^\sharp(w)})_{w \in S^*}$, obtained from the natural isomorphism of the following diagram

$$\begin{array}{ccccccc} & & (\cdot)^\natural_{T^*} \circ (\cdot)^\natural_T & & & & \\ & \curvearrowright & & \curvearrowleft & & & \\ \mathbf{Set}^T & \xrightarrow{(\cdot)^\natural_T} & \mathbf{Set}^{T^*} & \xrightarrow{\Delta_{\lambda_T}} & \mathbf{Set}^{T^{**}} & \xrightarrow{\Delta_{\varphi^*}} & \mathbf{Set}^{S^*} \\ & & \Downarrow \iota & & \parallel & & \\ & & & & \Delta_{\varphi^\sharp} & & \end{array}$$

(2) The isomorphism is $\iota_{\psi^\sharp \circ \varphi}^C = (\iota_{\psi^\sharp \circ \varphi(s)}^C: C_{\psi^\sharp \circ \varphi(s)} \longrightarrow C_{\psi^\sharp \circ \varphi(s)})_{s \in S}$, obtained from the natural isomorphism of the following diagram

$$\begin{array}{ccccccccc} & & (\cdot)^\natural_{U^*} \circ (\cdot)^\natural_U & & & & & & \\ & \curvearrowright & & \curvearrowleft & & & & & \\ \mathbf{Set}^U & \xrightarrow{(\cdot)^\natural_U} & \mathbf{Set}^{U^*} & \xrightarrow{\Delta_{\lambda_U}} & \mathbf{Set}^{U^{**}} & \xrightarrow{\Delta_{\psi^*}} & \mathbf{Set}^{T^*} & \xrightarrow{\Delta_\varphi} & \mathbf{Set}^S \\ & & \Downarrow \iota & & \parallel & & & & \\ & & & & \Delta_{\psi^\sharp} & & & & \end{array}$$

(3) It is enough to define, for two words $w, u \in S^*$, $(\kappa_\varphi^B)_{w,u}$, i.e., the component at (w, u) of κ_φ^B , as the isomorphism from $\text{Hom}(B_{\varphi^\sharp(w)}, B_{\varphi^\sharp(u)})$ to $\text{Hom}(B_{\varphi^*(w)}, B_{\varphi^*(u)})$

which sends a mapping $h: B_{\varphi^\sharp(w)} \longrightarrow B_{\varphi^\sharp(u)}$ to the composite mapping

$$B_{\varphi^*(w)} \xrightarrow{\iota_{\varphi^*(w)}^B} B_{\varphi^\sharp(w)} \xrightarrow{h} B_{\varphi^\sharp(u)} \xrightarrow{(\iota_{\varphi^*(u)}^B)^{-1}} B_{\varphi^*(u)},$$

where, we recall, $B_{\varphi^*(w)} = \prod_{i \in |w|} B_{\varphi(w_i)}$, and $B_{\varphi^*(u)} = \prod_{j \in |u|} B_{\varphi(u_j)}$. \square

Once defined the above auxiliary functors and natural transformations we prove in the following proposition that the polyderivators between signatures determine functors, in the opposite direction, from the category of algebras associated to the target signature to the category of algebras associated to the source signature. These functors will be the components of the morphism mapping of the contravariant pseudo-functor $\mathbf{Alg}_{\mathfrak{pd}}$ from $\mathbf{Sig}_{\mathfrak{pd}}$ to \mathbf{Cat} .

Proposition 54. *Let $\mathbf{d}: \Sigma \longrightarrow \Lambda$ be a morphism in $\mathbf{Sig}_{\mathfrak{pd}}$. Then there exists a functor $\mathbf{Alg}_{\mathfrak{pd}}(\mathbf{d}) = \mathbf{d}_{\mathfrak{pd}}^*$ from $\mathbf{Alg}(\Lambda)$ to $\mathbf{Alg}(\Sigma)$ defined as follows*

- (1) $\mathbf{d}_{\mathfrak{pd}}^*$ assigns to a Λ -algebra $\mathbf{B} = (B, G)$ the Σ -algebra $\mathbf{d}_{\mathfrak{pd}}^*(\mathbf{B}) = (B_\varphi, G^\mathbf{d})$, where $G^\mathbf{d}$ is $\kappa_\varphi^B \circ G_{\varphi^\sharp \times \varphi^\sharp}^\# \circ d$, obtained from

$$\begin{array}{ccc} \prod_{1 \times \mathcal{Q}_T} \Lambda & \xrightarrow{\eta_{\prod_{1 \times \mathcal{Q}_T} \Lambda}^{\mathbf{BT}}} & \mathbf{BTer}_T(\Lambda) \\ & \searrow G & \downarrow G^\# \\ & & \mathbf{BOP}_T(B) \end{array} \quad \text{as} \quad \begin{array}{ccc} \mathbf{BTer}_T(\Lambda)_{\varphi^\sharp \times \varphi^\sharp} & \xleftarrow{d} & \prod_{1 \times \mathcal{Q}_S} \Sigma \\ \downarrow G_{\varphi^\sharp \times \varphi^\sharp}^\# & & \\ \mathbf{BOP}_T(B)_{\varphi^\sharp \times \varphi^\sharp} & \xrightarrow{\kappa_\varphi^B} & \mathbf{BOP}_S(B_\varphi) \end{array}$$

- (2) $\mathbf{d}_{\mathfrak{pd}}^*$ assigns to a Λ -homomorphism f from \mathbf{B} to \mathbf{B}' the Σ -homomorphism $\mathbf{d}_{\mathfrak{pd}}^*(f) = f_\varphi$ from $\mathbf{d}_{\mathfrak{pd}}^*(\mathbf{B})$ to $\mathbf{d}_{\mathfrak{pd}}^*(\mathbf{B}')$.

Proof. It is obvious that $G^\mathbf{d}$, as defined, is an algebraic structure on B_φ .

Following this we prove that if f is a Λ -homomorphism from \mathbf{B} to \mathbf{B}' , then f_φ is a Σ -homomorphism from $\mathbf{d}_{\mathfrak{pd}}^*(\mathbf{B})$ to $\mathbf{d}_{\mathfrak{pd}}^*(\mathbf{B}')$.

Let $\sigma: w \longrightarrow s$ be an operation in Σ . Then in the following diagram

$$\begin{array}{ccccc} & & B_{\varphi^\sharp(w)} & \xrightarrow{(G_{\varphi^\sharp \times \varphi^\sharp}^\# \circ d)_{w,s}(\sigma)} & B_{\varphi^\sharp(s)} & & \\ & \nearrow \iota_{\varphi^*(w)}^B & \downarrow & & \downarrow & \searrow (\iota_{\varphi^*(s)}^B)^{-1} & \\ B_{\varphi_w} & \xrightarrow{f_{\varphi^\sharp(w)}} & B_{\varphi^\sharp(w)} & \xrightarrow{G_\sigma^\mathbf{d}} & B_{\varphi^\sharp(s)} & \xrightarrow{f_{\varphi^\sharp(s)}} & B_{\varphi_s} \\ & \downarrow f_{\varphi_w} & \downarrow f_{\varphi^\sharp(w)} & & \downarrow f_{\varphi^\sharp(s)} & & \downarrow f_{\varphi_s} \\ & \nearrow \iota_{\varphi^*(w)}^{B'} & B'_{\varphi^\sharp(w)} & \xrightarrow{(G'_{\varphi^\sharp \times \varphi^\sharp} \circ d)_{w,s}(\sigma)} & B'_{\varphi^\sharp(s)} & \searrow (\iota_{\varphi^*(s)}^{B'})^{-1} & \\ B'_{\varphi_w} & \xrightarrow{f_{\varphi^\sharp(w)}} & B'_{\varphi^\sharp(w)} & \xrightarrow{G'_\sigma^\mathbf{d}} & B'_{\varphi^\sharp(s)} & \xrightarrow{f_{\varphi^\sharp(s)}} & B'_{\varphi_s} \end{array}$$

the back face commutes because f is a morphism, the top and bottom faces commute by definition and the left and right faces commute because ι is natural. Hence, the front face commutes, but this means that f_φ is a Σ -homomorphism from $\mathbf{d}_{\mathfrak{pd}}^*(\mathbf{B})$ to $\mathbf{d}_{\mathfrak{pd}}^*(\mathbf{B}')$.

Finally we prove that $\mathbf{d}_{\mathfrak{p}\mathfrak{d}}^*$ is a functor. We restrict ourselves to verify that $\mathbf{d}_{\mathfrak{p}\mathfrak{d}}^*$ preserves compositions. Let $f: \mathbf{B} \longrightarrow \mathbf{B}'$ and $g: \mathbf{B}' \longrightarrow \mathbf{B}''$ be two Λ -homomorphisms and $s \in S$, then we have that

$$\begin{aligned}
(g \circ f)_{\varphi_s} &= (g \circ f)_{\varphi(s)} \\
&= (g \circ f)_{\varphi(s)_0} \times \cdots \times (g \circ f)_{\varphi(s)_{|\varphi(s)|-1}} \\
&= (g_{\varphi(s)_0} \circ f_{\varphi(s)_0}) \times \cdots \times (g_{\varphi(s)_{|\varphi(s)|-1}} \circ f_{\varphi(s)_{|\varphi(s)|-1}}) \\
&= (g_{\varphi(s)_0} \times \cdots \times g_{\varphi(s)_{|\varphi(s)|-1}}) \circ (f_{\varphi(s)_0} \times \cdots \times f_{\varphi(s)_{|\varphi(s)|-1}}) \\
&= g_{\varphi(s)} \circ f_{\varphi(s)} \\
&= g_{\varphi_s} \circ f_{\varphi_s}. \quad \square
\end{aligned}$$

Given a polyderivator $\mathbf{d}: \Sigma \longrightarrow \Lambda$, a Λ -algebra $\mathbf{B} = (B, G)$ and an operation $\sigma \in \Sigma_{w,s}$, if we agree that w is the word $(s_i)_{i \in m}$, that, for every $i \in m$, $\varphi(s_i)$ is the word $(t_{i,j})_{j \in n_i}$, and that $\varphi(s)$ is the word $(t_k)_{k \in p}$, then we have that $\varphi^\sharp(w)$ is the word

$$(t_{0,0}, \dots, t_{0,n_0-1}, \dots, t_{m-1,0}, \dots, t_{m-1,n_{m-1}-1})$$

and that $d(\sigma): \varphi^\sharp(w) \longrightarrow \varphi(s)$ is a family of terms $P = (P_0, \dots, P_{p-1})$ such that, for every $k \in p$, $P_k: \varphi^\sharp(w) \longrightarrow t_k$. Thus the realization of $d(\sigma)$ in \mathbf{B} , $G_{\varphi^\sharp \times \varphi^\sharp}^\sharp(P)$, is the term operation $P^\mathbf{B} = \langle P_0^\mathbf{B}, \dots, P_{p-1}^\mathbf{B} \rangle$ of type

$$B_{t_{0,0}} \times \cdots \times B_{t_{0,n_0-1}} \times \cdots \times B_{t_{m-1,0}} \times \cdots \times B_{t_{m-1,n_{m-1}-1}} \longrightarrow B_{t_0} \times \cdots \times B_{t_{p-1}}$$

that by composition with the isomorphism from B_{φ_w} to $B_{\varphi^\sharp(w)}$ provides the operation $G^\mathbf{d}(\sigma)$

$$\begin{array}{c}
(B_{t_{0,0}} \times \cdots \times B_{t_{0,n_0-1}}) \times \cdots \times (B_{t_{m-1,0}} \times \cdots \times B_{t_{m-1,n_{m-1}-1}}) \\
\downarrow \iota_{(\varphi(s_0), \dots, \varphi(s_{m-1}))}^\mathbf{B} \\
B_{t_{0,0}} \times \cdots \times B_{t_{0,n_0-1}} \times \cdots \times B_{t_{m-1,0}} \times \cdots \times B_{t_{m-1,n_{m-1}-1}} \\
\downarrow P^\mathbf{B} \\
B_{t_0} \times \cdots \times B_{t_{p-1}}
\end{array}$$

It is now when we can prove that the contravariant functor \mathbf{Alg} from \mathbf{Sig} to \mathbf{Cat} , defined in the second section, can be lifted to a contravariant pseudo-functor $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$ from $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to \mathbf{Cat} , and, as was the case there, by applying to this contravariant pseudo-functor the construction of Ehresmann-Grothendieck, we get the new category $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$.

Proposition 55. *There exists a contravariant pseudo-functor $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$ from $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to the 2-category \mathbf{Cat} given by the following data*

- (1) *The object mapping of $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$ is that which sends a signature Σ to $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}(\Sigma) = \mathbf{Alg}(\Sigma)$.*
- (2) *The morphism mapping of $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$ is that which sends a polyderivator \mathbf{d} from Σ to Λ to $\mathbf{d}_{\mathfrak{p}\mathfrak{d}}^*: \mathbf{Alg}(\Lambda) \longrightarrow \mathbf{Alg}(\Sigma)$.*
- (3) *For every $\mathbf{d}: \Sigma \longrightarrow \Lambda$ and $\mathbf{e}: \Lambda \longrightarrow \Omega$, the natural isomorphism $\gamma^{\mathbf{d}, \mathbf{e}}$ from $\mathbf{e}_{\mathfrak{p}\mathfrak{d}}^* \circ \mathbf{d}_{\mathfrak{p}\mathfrak{d}}^*$ to $(\mathbf{e} \circ \mathbf{d})_{\mathfrak{p}\mathfrak{d}}^*$ is that which is defined, for every Ω -algebra \mathbf{C} , as the isomorphism $\iota_{\psi^* \circ \varphi}^{\mathbf{C}}$.*
- (4) *For every Σ , the natural isomorphism ν^Σ from $\text{Id}_{\mathbf{Alg}(\Sigma)}$ to $(\check{\eta}_S, \eta_\Sigma^{\mathbf{B}_S})_{\mathfrak{p}\mathfrak{d}}^*$ is that which is defined, for every Σ -algebra \mathbf{A} , as the canonical isomorphism $\delta_S^{\mathbf{A}}: \mathbf{A} \longrightarrow (A_{(s)})_{s \in S}$.*

Proof. Given $\mathbf{d}: \Sigma \longrightarrow \mathbf{A}$ and $\mathbf{e}: \mathbf{A} \longrightarrow \mathbf{O}$, we prove that

$$\iota_{\psi^* \circ \varphi}^C: (C_{\psi \varphi}, H^{\mathbf{e}^{\mathbf{d}}}) \longrightarrow (C_{\psi^{\#} \circ \varphi}, H^{(\psi^{\#} \circ \varphi, e_{\varphi^{\#} \times \psi^{\#}}^{\# \circ \mathbf{d}})})$$

is an isomorphism of Σ -algebras, and for this it is enough to prove that it is a morphism, because $\iota_{\psi^* \circ \varphi}^C$ is a bijection.

But the following diagram commutes

$$\begin{array}{ccc} & \Sigma & \\ H^{\mathbf{e}^{\mathbf{d}}} \swarrow & & \searrow H^{\mathbf{e}^{\mathbf{od}}} \\ \text{BOP}_{S^*}(C_{\psi \varphi}) & \xrightarrow{\text{Op}_B(\iota_{\psi^* \circ \varphi}^C)} & \text{BOP}_S(C_{\psi^{\#} \circ \varphi}) \end{array}$$

where $\text{Op}_B(\iota_{\psi^* \circ \varphi}^C)$ is the isomorphism from $\text{BOP}_{S^*}(C_{\psi \varphi})$ to $\text{BOP}_S(C_{\psi^{\#} \circ \varphi})$ induced by the isomorphism $\iota_{\psi^* \circ \varphi}^C$, because, on the one hand, we have that

$$\begin{aligned} H^{\mathbf{e}^{\mathbf{d}}} &= \kappa_{\varphi}^{C_{\psi}} \circ (H^{\mathbf{e}})_{\varphi^{\#} \times \psi^{\#}}^{\#} \circ d \\ &= \kappa_{\varphi}^{C_{\psi}} \circ (\kappa_{\psi}^C \circ H_{\psi^{\#} \times \psi^{\#}}^{\#} \circ e)_{\varphi^{\#} \times \psi^{\#}}^{\#} \circ d \\ &= \kappa_{\varphi}^{C_{\psi}} \circ (\kappa_{\psi}^C \circ H_{\psi^{\#} \times \psi^{\#}}^{\#} \circ e^{\#})_{\varphi^{\#} \times \psi^{\#}} \circ d \\ &= \kappa_{\varphi}^{C_{\psi}} \circ (\kappa_{\psi}^C)_{\varphi^{\#} \times \psi^{\#}} \circ (H_{\psi^{\#} \times \psi^{\#}}^{\#})_{\varphi^{\#} \times \psi^{\#}} \circ e_{\varphi^{\#} \times \psi^{\#}}^{\#} \circ d \\ &= \kappa_{\varphi}^{C_{\psi}} \circ (\kappa_{\psi}^C)_{\varphi^{\#} \times \psi^{\#}} \circ H_{(\psi^{\#} \circ \varphi)^{\#} \times (\psi^{\#} \circ \varphi)^{\#}}^{\#} \circ e_{\varphi^{\#} \times \psi^{\#}}^{\#} \circ d, \end{aligned}$$

on the other hand, that

$$\begin{aligned} H^{\mathbf{e}^{\mathbf{od}}} &= H^{(\psi^{\#} \circ \varphi, e_{\varphi^{\#} \times \psi^{\#}}^{\# \circ \mathbf{d}})} \\ &= \kappa_{\psi^{\#} \circ \varphi}^C \circ H_{(\psi^{\#} \circ \varphi)^{\#} \times (\psi^{\#} \circ \varphi)^{\#}}^{\#} \circ e_{\varphi^{\#} \times \psi^{\#}}^{\#} \circ d, \end{aligned}$$

and, lastly, that the following diagram commutes

$$\begin{array}{ccc} (\text{BOP}_U(C)_{\psi^{\#} \times \psi^{\#}})_{\varphi^{\#} \times \psi^{\#}} & & \\ \downarrow (\kappa_{\psi}^C)_{\varphi^{\#} \times \psi^{\#}} & \searrow \kappa_{\psi^{\#} \circ \varphi}^C & \\ \text{BOP}_T(C_{\psi})_{\varphi^{\#} \times \psi^{\#}} & & \\ \downarrow \kappa_{\varphi}^{C_{\psi}} & & \\ \text{BOP}_{S^*}(C_{\psi \varphi}) & \xrightarrow{\text{Op}_B(\iota_{\psi^* \circ \varphi}^C)} & \text{BOP}_S(C_{\psi^{\#} \circ \varphi}) \end{array}$$

□

Definition 32. The category $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$ of algebras and morphisms, obtained by applying the Ehresmann-Grothendieck construction to the contravariant pseudo-functor $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$, is $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}} = \int^{\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}} \mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$.

Therefore the category $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$ has as objects the pairs (Σ, \mathbf{A}) , with Σ a signature and \mathbf{A} a Σ -algebra, and as morphisms from (Σ, \mathbf{A}) to (Λ, \mathbf{B}) , the pairs (\mathbf{d}, h) , with \mathbf{d} a polyderivor from Σ to Λ and h a Σ -homomorphism from \mathbf{A} to $\mathbf{d}_{\mathfrak{p}\mathfrak{d}}^*(\mathbf{B})$. Hence,

for every $(w, s) \in S^* \times S$ and $\sigma \in \Sigma_{w,s}$ the following diagram commutes

$$\begin{array}{ccc} A_w & \xrightarrow{h_w} & (\prod_{j \in n_0} B_{t_{0,j}}) \times \cdots \times (\prod_{j \in n_{m-1}} B_{t_{m-1,j}}) \\ F_\sigma \downarrow & & \downarrow G^{\mathbf{d}}(\sigma) \\ A_s & \xrightarrow{h_s} & \prod_{k \in p} B_{t_k} \end{array}$$

where we have agreed that w is the word $(s_i)_{i \in m}$, that, for every $i \in m$, $\varphi(s_i)$ is the word $(t_{i,j})_{j \in n_i}$, and that $\varphi(s)$ is the word $(t_k)_{k \in p}$.

Example. Let Σ be a signature, $p \in \mathbb{N}$, and $\mathbf{d} = (\varphi, d)$ the polyderivator from Σ into itself, where

- (1) $\varphi: S \longrightarrow S^*$ is the mapping which sends $s \in S$ to the word $\lambda_{\mu \in p}(s)$ and,
- (2) For $(w, s) \in S^* \times S$, $d_{w,s}$ is the mapping from $\Sigma_{w,s}$ to $T_\Sigma(\downarrow \varphi^\sharp(w))_s^p$ which sends $\sigma \in \Sigma_{w,s}$ to

$$(\sigma(v_0^{w_0}, v_p^{w_1}, \dots, v_{(|w|-1)p}^{w_{|w|-1}}), \dots, \sigma(v_{p-1}^{w_0}, v_{2p-1}^{w_1}, \dots, v_{|w|p-1}^{w_{|w|-1}})),$$

in $T_\Sigma(\downarrow \varphi^\sharp(w))_s^p$.

Then, for the polyderivator \mathbf{d} and two Σ -algebras \mathbf{A} and \mathbf{B} , we have that $(\mathbf{d}, \langle h^\mu \rangle_{\mu \in p})$, where, for every $\mu \in p$, $h^\mu = (h_s^\mu)_{s \in S}$ is a Σ -homomorphism from \mathbf{A} to \mathbf{B} , is a morphism from (Σ, \mathbf{A}) to (Σ, \mathbf{B}) , because $\mathbf{d}_{\mathbf{p}\mathbf{d}}^*(\mathbf{B}) = \mathbf{B}^p$.

The following example of morphism between algebras (stated using the terminology of Fujiwara) although redundant, since it is a particular instance of the preceding one, is provided because, being referred to single-sorted signatures and algebras, it is by far less troublesome and more easily graspable.

Example. Let $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ be a single-sorted signature, $\Phi = \{\varphi_\mu \mid \mu \in m\}$, and $P = (P^n)_{n \in \mathbb{N}}$ the family defined, for every natural number $n \in \mathbb{N}$, as follows

$$P^n \begin{cases} \Phi \times \Sigma_n \longrightarrow T_\Sigma(\Phi \times \downarrow v_n) \\ (\varphi_\mu, \sigma) \longmapsto \sigma(\varphi_\mu(v_0), \dots, \varphi_\mu(v_{n-1})) \end{cases}$$

Then, for the polyderivator \mathbf{d} associated to (Φ, P) and two Σ -algebras \mathbf{A} and \mathbf{B} , we have that $(\mathbf{d}, \langle h_\mu \rangle_{\mu \in p})$, where, for every $\mu \in p$, h_μ is a Σ -homomorphism from \mathbf{A} to \mathbf{B} , is a morphism from (Σ, \mathbf{A}) to (Σ, \mathbf{B}) , because $\mathbf{d}_{\mathbf{p}\mathbf{d}}^*(\mathbf{B}) = \mathbf{B}^m$.

Example. Let $\Sigma = (\Sigma_n)_{n \in \mathbb{N}}$ be a single-sorted signature such that $\Sigma_2 = \{+, -, \cdot\}$ and $\Sigma_n = \emptyset$, if $n \neq 2$, $\Phi = \{\varphi_{\mu,\nu} \mid (\mu, \nu) \in 2 \times 2\}$, and $P = (P^n)_{n \in \mathbb{N}}$ the family defined, for $n \neq 2$, as the unique mapping from \emptyset to $T_\Sigma(\Phi \times \downarrow v_n)$, and, for $n = 2$, as follows

- (1) $P_{\varphi_{\mu,\nu},+}^2 = \varphi_{\mu,\nu}(v_0) + \varphi_{\mu,\nu}(v_1)$.
- (2) $P_{\varphi_{\mu,\nu},-}^2 = \varphi_{\mu,\nu}(v_0) - \varphi_{\mu,\nu}(v_1)$.
- (3) $P_{\varphi_{\mu,\nu},\cdot}^2 = \sum_{\lambda=0}^1 \varphi_{\mu,\lambda}(v_0) \cdot \varphi_{\lambda,\nu}(v_1)$.

Then, for the polyderivator \mathbf{d} associated to (Φ, P) and two rings \mathbf{A} and \mathbf{B} , we have that $(\mathbf{d}, \begin{pmatrix} f & \kappa_0 \\ d & g \end{pmatrix})$, where f and g are two ring homomorphisms from \mathbf{A} to \mathbf{B} , κ_0 is the mapping from A to B that is constantly 0, and d an (f, g) -derivation from \mathbf{A} to \mathbf{B} , is a morphism from (Σ, \mathbf{A}) to (Σ, \mathbf{B}) , because $\mathbf{d}_{\mathbf{p}\mathbf{d}}^*(\mathbf{B}) = \mathbf{B}^{2 \times 2}$, the matrix ring of degree 2 over \mathbf{B} .

Example. Taking as set of sorts S a set with two elements $\{s, v\}$, where the sort s is for “scalar” and the sort v for “vector”, and as S -sorted signature Σ that one

adequate for the concept of vectorial space (with the scalar field variable), we have that for a \mathbf{K} -vectorial space \mathbf{V} and a \mathbf{K}' -vectorial space \mathbf{V}' , if f is a homomorphism from the field \mathbf{K} to the field \mathbf{K}' and g is an f -semilineal morphism from the \mathbf{K} -vectorial space \mathbf{V} to the \mathbf{K}' -vectorial space \mathbf{V}' , then $(\text{id}_{\Sigma}, (f, g))$ is a morphism from $(\Sigma, (\mathbf{K}, \mathbf{V}))$ to $(\Sigma, (\mathbf{K}', \mathbf{V}'))$.

More examples like this one can be obtained, e.g., from the concept of \mathbf{R} -module, for a ring \mathbf{R} , from that of \mathbf{G} -set, for a group \mathbf{G} , from that of \mathbf{M} -set, for a monoid \mathbf{M} , or from metric or pseudo-metric spaces.

Additional examples related to computer sciences can be found in [27].

The contravariant pseudo-functor Alg_{pd} is not only useful to construct the category \mathbf{Alg}_{pd} . Actually, as we prove in that which follows, it together with a pseudo-functor Ter_{pd} from \mathbf{Sig}_{pd} to \mathbf{Cat} , and a pseudo-extranatural transformation (Tr, θ) (from a pseudo-functor on $\mathbf{Sig}_{\text{pd}}^{\text{op}} \times \mathbf{Sig}_{\text{pd}}$ to \mathbf{Cat} , induced by Alg_{pd} and Ter_{pd} , to the functor, between the same categories, constantly \mathbf{Set}), determine an institution $\mathfrak{Im}_{\text{pd}} = (\mathbf{Sig}_{\text{pd}}, \text{Alg}_{\text{pd}}, \text{Ter}_{\text{pd}}, (\text{Tr}, \theta))$ on \mathbf{Set} , the so-called *many-sorted term institution of Fujiwara*.

We define next some auxiliary functors and natural transformations that we will use afterwards to prove, on the one hand, that there exists a pseudo-functor Ter_{pd} from the category \mathbf{Sig}_{pd} to the 2-category \mathbf{Cat} , which generalizes the pseudo-functor Ter from the category \mathbf{Sig} to the 2-category \mathbf{Cat} , and, on the other hand, that the category \mathbf{Alg}_{pd} has coproducts.

Proposition 56. *Let S be a set of sorts. Then we have that*

- (1) *There exists a compression functor $(\cdot)^{\dagger S}$ from \mathbf{Set}^{S^*} to \mathbf{Set}^S , left adjoint to the expansion functor $(\cdot)^{\natural S}$, defined, for every S^* -sorted set C and $s \in S$, as follows*

$$C_s^{\dagger S} = \bigcup_{\substack{w \in S^* \\ w^{-1}[s] \neq \emptyset}} (C_w \times \{w\} \times w^{-1}[s]),$$

and, for every S^* -mapping $f: C \longrightarrow C'$, $s \in S$ and (c, w, i) in $C_s^{\dagger S}$, as follows

$$f_s^{\dagger S}(c, w, i) = (f_w(c), w, i).$$

- (2) *From the contravariant functor $\text{MSet} \circ \text{T}_{\star}^{\text{op}}$, from \mathbf{Set} to \mathbf{Cat} , to the contravariant functor MSet between the same categories, there exists a natural transformation $(\cdot)^{\dagger}$ which sends a set S to the compression functor $(\cdot)^{\dagger S}$ from \mathbf{Set}^{S^*} to \mathbf{Set}^S .*
- (3) *There exists a natural isomorphism ζ_S from the functor $(\cdot)^{\dagger S} \circ (\cdot)^{\dagger S^*}$ to the functor $(\cdot)^{\dagger S} \circ \prod_{\lambda_S}$.*

Proof. We restrict ourselves to prove the first and third part of the proposition.

(1) For every S^* -sorted set C and S -sorted set A , there exists a natural isomorphism $\theta^{\dagger \natural}: \text{Hom}(C^{\dagger S}, A) \cong \text{Hom}(C, A^{\natural S})$ which assigns to an S -sorted mapping $f: C^{\dagger S} \longrightarrow A$ the S^* -sorted mapping $\theta^{\dagger \natural}(f)$, defined, for every $w \in S^*$ and $c \in C_w$, as $\theta^{\dagger \natural}(f)_w(c) = (f_{w_i}(c, w, i))_{i \in |w|}$.

Reciprocally, if $g: C \longrightarrow A^{\natural S}$ is an S^* -sorted mapping, then $(\theta^{\dagger \natural})^{-1}(g)$ is the S -sorted mapping defined, for every $s \in S$ and $(c, w, i) \in C_s^{\dagger S}$, as $(\theta^{\dagger \natural})^{-1}(g)_s(c, w, i) = g_w(c)_i$.

(3) By Proposition 53, the functors $(\cdot)^{\natural S^*} \circ (\cdot)^{\natural S}$ and $\Delta_{\lambda_S} \circ (\cdot)^{\natural S}$ are isomorphic. Furthermore, $(\cdot)^{\dagger S} \circ (\cdot)^{\dagger S^*}$ is left adjoint to $(\cdot)^{\natural S^*} \circ (\cdot)^{\natural S}$ and $(\cdot)^{\dagger S} \circ \prod_{\lambda_S}$ is left adjoint to $\Delta_{\lambda_S} \circ (\cdot)^{\natural S}$, thus the functors $(\cdot)^{\dagger S} \circ (\cdot)^{\dagger S^*}$ and $(\cdot)^{\dagger S} \circ \prod_{\lambda_S}$ are isomorphic. We denote such a natural isomorphism by ζ_S , and, to simplify the notation, we will write ζ^C instead of $(\zeta_S)_C$ for the component of ζ_S at C . \square

If $\varphi: S \longrightarrow T^*$ is a mapping, then from the adjunctions $\coprod_{\varphi} \dashv \Delta_{\varphi}$ and $(\cdot)^{\dagger T} \dashv (\cdot)^{\natural T}$, we get the adjunction $\coprod_{\varphi}^{\dagger} \dashv \Delta_{\varphi}^{\natural}$ as reflected in the following diagram

$$\begin{array}{ccc}
 & \mathbf{Set}^S & \\
 \coprod_{\varphi}^{\dagger} \swarrow & & \searrow \Delta_{\varphi} \\
 \mathbf{Set}^T & & \mathbf{Set}^{T^*} \\
 \Delta_{\varphi}^{\natural} \swarrow & & \searrow \coprod_{\varphi} \\
 & \mathbf{Set}^{T^*} & \\
 \top & & \\
 (\cdot)^{\dagger T} \swarrow & & \searrow (\cdot)^{\natural T}
 \end{array}$$

where, to simplify the notation, we have written $\coprod_{\varphi}^{\dagger}$ instead of $(\cdot)^{\dagger T} \circ \coprod_{\varphi}$ and $\Delta_{\varphi}^{\natural}$ instead of $\Delta_{\varphi} \circ (\cdot)^{\natural T}$. Furthermore, we agree that $\theta_{\varphi}^{\dagger \natural}$, $\eta_{\varphi}^{\dagger \natural}$, and $\varepsilon_{\varphi}^{\dagger \natural}$ denote, respectively, the natural isomorphism, the unit, and the counit of this composite adjunction.

Since it could be of some help, next we recall the explicit definitions of $\coprod_{\varphi}^{\dagger}$, $\Delta_{\varphi}^{\natural}$, and $\theta_{\varphi}^{\dagger \natural}$.

- (1) The functor $\coprod_{\varphi}^{\dagger}$ assigns to an S -sorted set A the T -sorted set $\coprod_{\varphi}^{\dagger} A$ whose t -th coordinate, for $t \in T$, is

$$(\coprod_{\varphi}^{\dagger} A)_t = \bigcup_{\substack{x \in T^* \\ x^{-1}[t] \neq \emptyset}} ((\bigcup_{s \in \varphi^{-1}[x]} (A_s \times \{s\})) \times \{x\} \times x^{-1}[t]),$$

therefore $(\coprod_{\varphi}^{\dagger} A)_t$ has as members the ordered quadruples (a, s, x, j) such that $a \in A_s$, $\varphi(s) = x$ and $\varphi(s)_j = t$.

- (2) The functor $\Delta_{\varphi}^{\natural}$ assigns to a T -sorted set B the S -sorted set $\Delta_{\varphi}^{\natural} B = (B_{\varphi(s)})_{s \in S}$, where, we recall, $B_{\varphi(s)} = \prod_{i \in |\varphi(s)|} B_{\varphi(s)_i}$.
- (3) For an S -sorted set A and a T -sorted set B , the natural isomorphism $\theta_{\varphi}^{\dagger \natural}$ of the adjunction $\coprod_{\varphi}^{\dagger} \dashv \Delta_{\varphi}^{\natural}$ sends a T -sorted mapping $f: \coprod_{\varphi}^{\dagger} A \longrightarrow B$ to the S -sorted mapping $\theta_{\varphi}^{\dagger \natural}(f): A \longrightarrow \Delta_{\varphi}^{\natural} B$ that, for $s \in S$ and $a \in A_s$, is defined as follows

$$\theta_{\varphi}^{\dagger \natural}(f)_s \begin{cases} A_s & \longrightarrow B_{\varphi(s)} \\ a & \longmapsto (f_{\varphi(s)_i}(a, s, \varphi(s), i))_{i \in |\varphi(s)|} \end{cases}$$

and $(\theta_{\varphi}^{\dagger \natural})^{-1}$ sends an S -sorted mapping $g: A \longrightarrow \Delta_{\varphi}^{\natural} B$ to the T -sorted mapping $(\theta_{\varphi}^{\dagger \natural})^{-1}(g): \coprod_{\varphi}^{\dagger} A \longrightarrow B$ that, for $t \in T$, is defined as follows

$$(\theta_{\varphi}^{\dagger \natural})^{-1}(g)_t \begin{cases} (\coprod_{\varphi}^{\dagger} A)_t & \longrightarrow B_t \\ (a, s, \varphi(s), i) & \longmapsto g_s(a)_i \end{cases}$$

What we want to establish now is that the category $\mathbf{Alg}_{\mathfrak{p}^{\mathfrak{d}}}$ has coproducts and for this we begin by proving that, for every polyderivator $\mathbf{d}: \Sigma \longrightarrow \Lambda$, the functor $\mathbf{d}_{\mathfrak{p}^{\mathfrak{d}}}^*$ from $\mathbf{Alg}(\Lambda)$ to $\mathbf{Alg}(\Sigma)$ has a left adjoint $\mathbf{d}_{\mathfrak{p}^{\mathfrak{d}}}^{\mathfrak{p}^{\mathfrak{d}}}$.

Proposition 57. *Let $\mathbf{d}: \Sigma \longrightarrow \Lambda$ be a polyderivator. Then there exists a functor $\mathbf{d}_{\mathfrak{p}^{\mathfrak{d}}}^{\mathfrak{p}^{\mathfrak{d}}}$ from $\mathbf{Alg}(\Sigma)$ to $\mathbf{Alg}(\Lambda)$ that is left adjoint to the functor $\mathbf{d}_{\mathfrak{p}^{\mathfrak{d}}}^*$ from $\mathbf{Alg}(\Lambda)$ to $\mathbf{Alg}(\Sigma)$.*

Proof. We restrict ourselves to define the action of $\mathbf{d}_{\mathfrak{p}^{\mathfrak{d}}}^{\mathfrak{p}^{\mathfrak{d}}}$ on the objects (because being the remaining details like those of Proposition 11, although more cumbersome, they can be left to the reader). Let \mathbf{A} be a Σ -algebra. Then $\mathbf{d}_{\mathfrak{p}^{\mathfrak{d}}}^{\mathfrak{p}^{\mathfrak{d}}}(\mathbf{A})$ is the Λ -algebra

defined as $\mathbf{T}_\Lambda(\coprod_\varphi^\dagger A)/\bar{R}^\mathbf{A}$, where $\bar{R}^\mathbf{A}$ is the congruence on $\mathbf{T}_\Lambda(\coprod_\varphi^\dagger A)$ generated by the T -sorted relation $R^\mathbf{A}$, defined, for every $t \in T$, as

$$R_t^\mathbf{A} = \left\{ \left((F_\sigma^\mathbf{A}(a_i \mid i \in |w|), s, \varphi(s), j), d(\sigma)_j(\mathbf{a})) \mid \begin{array}{l} j \in \varphi(s)^{-1}[t], w \in S^*, \\ s \in S, \sigma \in \Sigma_{w,s}, a \in A_w \end{array} \right. \right\},$$

being \mathbf{a} the matrix

$$\mathbf{a} = \begin{pmatrix} (a_0, w_0, \varphi(w_0), 0) & \cdots & (a_0, w_0, \varphi(w_0), |\varphi(w_0)|-1) \\ \vdots & \ddots & \vdots \\ (a_{|w|-1}, w_{|w|-1}, \varphi(w_{|w|-1}), 0) & \cdots & (a_{|w|-1}, w_{|w|-1}, \varphi(w_{|w|-1}), |\varphi(w_{|w|-1})|-1) \end{pmatrix},$$

and $d(\sigma)_j(\mathbf{a})$ the result of replacing the variables in the term $d(\sigma)_j$ with the entries in the matrix \mathbf{a} (recall that, for $\sigma \in \Sigma_{w,s}$, we have agreed that $d(\sigma) = d_{w,s}(\sigma)$, where $d_{w,s}(\sigma) \in \mathbf{T}_\Lambda(\downarrow\varphi^\sharp(w))_{\varphi(s)}$, hence, for every $j \in |\varphi(s)|$, $d(\sigma)_j \in \mathbf{T}_\Lambda(\downarrow\varphi^\sharp(w))_{\varphi(s)_j}$). \square

Proposition 58. *The category $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$ has coproducts.*

Proof. The category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ has coproducts. For every signature Σ , the category $\mathbf{Alg}(\Sigma)$ has coproducts. The functor $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$ is locally reversible. Therefore, by a particular case of Proposition 3, the category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ has coproducts. \square

Our next goal is to state that every polyderivator induces a functor between the associated categories of terms as was the case for the signature morphisms.

Proposition 59. *Let $\mathbf{d}: \Sigma \longrightarrow \Lambda$ be a polyderivator. Then there exists a functor $\mathbf{d}_\circ^{\mathfrak{p}\mathfrak{d}}$ from $\mathbf{Ter}(\Sigma)$ to $\mathbf{Ter}(\Lambda)$ defined as follows*

- (1) $\mathbf{d}_\circ^{\mathfrak{p}\mathfrak{d}}$ sends an S -sorted set X to the T -sorted set $\mathbf{d}_\circ^{\mathfrak{p}\mathfrak{d}}(X) = \coprod_\varphi^\dagger X$.
- (2) $\mathbf{d}_\circ^{\mathfrak{p}\mathfrak{d}}$ sends a morphism P from X to Y in $\mathbf{Ter}(\Sigma)$ to the morphism $\mathbf{d}_\circ^{\mathfrak{p}\mathfrak{d}}(P) = (\theta_\varphi^{\dagger\sharp})^{-1}(\eta_X^\mathbf{d} \circ P)$ from $\coprod_\varphi^\dagger X$ to $\coprod_\varphi^\dagger Y$, where $\theta_\varphi^{\dagger\sharp}$ is the natural isomorphism of the adjunction $\coprod_\varphi^\dagger \dashv \Delta_\varphi^\natural$, $\eta_X^\mathbf{d}$ the Σ -homomorphism from $\mathbf{T}_\Sigma(X)$ to $\Delta_\varphi^\natural(\mathbf{T}_\Lambda(\coprod_\varphi^\dagger X))$ that extends the S -sorted mapping $\Delta_\varphi^\natural(\eta_{\coprod_\varphi^\dagger X}) \circ (\eta_\varphi^{\dagger\sharp})_X$ from X to $\Delta_\varphi^\natural(\mathbf{T}_\Lambda(\coprod_\varphi^\dagger X))$, as in the following commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathbf{T}_\Sigma(X) \\ \downarrow (\eta_\varphi^{\dagger\sharp})_X & & \downarrow \eta_X^\mathbf{d} = (\Delta_\varphi^\natural(\eta_{\coprod_\varphi^\dagger X}) \circ (\eta_\varphi^{\dagger\sharp})_X)^\sharp \\ \Delta_\varphi^\natural \coprod_\varphi^\dagger X & \xrightarrow{\Delta_\varphi^\natural(\eta_{\coprod_\varphi^\dagger X})} & \Delta_\varphi^\natural(\mathbf{T}_\Lambda(\coprod_\varphi^\dagger X)) \end{array}$$

and $\eta_\varphi^{\dagger\sharp}$ the unit of the adjunction $\coprod_\varphi^\dagger \dashv \Delta_\varphi^\natural$.

Proof. The proof is structurally identical to that of Proposition 15. However, we remark that besides it, there is another alternative proof founded on the fact that, for every term $P: X \longrightarrow Y$, the term $\mathbf{d}_\circ^{\mathfrak{p}\mathfrak{d}}(P): \coprod_\varphi^\dagger X \longrightarrow \coprod_\varphi^\dagger Y$ is the composition of the morphisms in the following diagram

$$\begin{array}{ccc} \coprod_\varphi^\dagger Y & \xrightarrow{\coprod_\varphi^\dagger P} & \coprod_\varphi^\dagger \mathbf{T}_\Sigma(X) \xrightarrow{\coprod_\varphi^\dagger \eta_X^\mathbf{d}} \coprod_\varphi^\dagger \Delta_\varphi^\natural(\mathbf{T}_\Lambda(\coprod_\varphi^\dagger X)) \\ & & \downarrow (\varepsilon_\varphi^{\dagger\sharp})_{\mathbf{T}_\Lambda(\coprod_\varphi^\dagger X)} \\ & & \mathbf{T}_\Lambda(\coprod_\varphi^\dagger X) \end{array}$$

where $(\varepsilon_\varphi^{\dagger\sharp})_{T_\Lambda(\coprod_\varphi^\dagger X)}$ is the value at $T_\Lambda(\coprod_\varphi^\dagger X)$ of the counit of the adjunction $\coprod_\varphi^\dagger \dashv \Delta_\varphi^\sharp$. \square

Remark. In the last proposition, $(\eta_\varphi^{\dagger\sharp})_X$, the value at X of the unit of the adjunction $\coprod_\varphi^\dagger \dashv \Delta_\varphi^\sharp$, in its s -th coordinate, assigns to $x \in X_s$, the family

$$((x, s, \varphi(s), 0), \dots, (x, s, \varphi(s), |\varphi(s)| - 1)).$$

We can say, informally, that to a variable $x \in X_s$ corresponds in $\coprod_\varphi^\dagger X$ a family of variables of the form $(x, s, \varphi(s), i)$, with sorts $\varphi(s)_i$, and that, for a morphism $P: X \longrightarrow Y$ in $\mathbf{Ter}(\Sigma)$ and $(y, s, \varphi(s), i) \in (\coprod_\varphi^\dagger Y)_t$, $\mathbf{d}_\diamond^{\text{pd}}(P)_{\varphi(s)_i}(y, s, \varphi(s), i)$ is the term for Λ obtained by replacing, recursively, in $P_s(y)$ the formal operations $\sigma: w \longrightarrow s$ by the families of formal operations $d(\sigma): \varphi^\sharp(w) \longrightarrow \varphi(s)$ and the variables $x \in X_s$ by families of variables $(x, s, \varphi(s), j)_{j \in |\varphi(s)|}$.

Before we state that the above construction can be lifted to a pseudo-functor from the category \mathbf{Sig}_{pd} to the 2-category \mathbf{Cat} , we point out that the relation of satisfaction is also invariant under polyderivator change, i.e., that for every polyderivator $\mathbf{d}: \Sigma \longrightarrow \Lambda$, if (P, Q) is a Σ -equation of type (X, Y) and \mathbf{A} a Λ -algebra, then

$$\mathbf{d}_{\text{pd}}^*(\mathbf{A}) \models_{X, Y}^\Sigma (P, Q) \text{ iff } \mathbf{A} \models_{\coprod_\varphi^\dagger X, \coprod_\varphi^\dagger Y}^\Lambda (\mathbf{d}_\diamond^{\text{pd}}(P), \mathbf{d}_\diamond^{\text{pd}}(Q)).$$

This follows from the invariant character under signature change through the polyderivators of the realization of terms as term operations in arbitrary, but fixed, algebras, as stated in the following

Proposition 60. *Let $\mathbf{d}: \Sigma \longrightarrow \Lambda$ be a polyderivator. Then, for every Λ -algebra \mathbf{A} and term $P: X \longrightarrow Y$ for Σ of type (X, Y) , the following diagram commutes*

$$\begin{array}{ccc} (A_\varphi)_X & \xrightarrow{P^{\mathbf{d}_{\text{pd}}^*(\mathbf{A})}} & (A_\varphi)_Y \\ \uparrow (\theta_\varphi^{\dagger\sharp})_{X, \mathbf{A}} & & \uparrow (\theta_\varphi^{\dagger\sharp})_{Y, \mathbf{A}} \\ A_{\coprod_\varphi^\dagger X} & \xrightarrow{\mathbf{d}_\diamond^{\text{pd}}(P)^\mathbf{A}} & A_{\coprod_\varphi^\dagger Y} \end{array}$$

where, to simplify the notation, we have written A_φ instead of the more accurate $\Delta_\varphi^\sharp A$.

Proof. The proof is analogous to that of Proposition 16. \square

It is now when we can state that the pseudo-functor \mathbf{Ter} from \mathbf{Sig} to the 2-category \mathbf{Cat} , defined in the second section, can be lifted up to a pseudo-functor \mathbf{Ter}_{pd} from \mathbf{Sig}_{pd} to the 2-category \mathbf{Cat} .

Proposition 61. *There exists a pseudo-functor \mathbf{Ter}_{pd} from \mathbf{Sig}_{pd} to the 2-category \mathbf{Cat} given by the following data*

- (1) *The object mapping of \mathbf{Ter}_{pd} is that which sends a signature Σ to $\mathbf{Ter}_{\text{pd}}(\Sigma) = \mathbf{Ter}(\Sigma)$.*
- (2) *The morphism mapping of \mathbf{Ter}_{pd} is that which sends a signature morphism $\mathbf{d}: \Sigma \longrightarrow \Lambda$ to $\mathbf{d}_\diamond^{\text{pd}}: \mathbf{Ter}(\Sigma) \longrightarrow \mathbf{Ter}(\Lambda)$.*
- (3) *For $\mathbf{d}: \Sigma \longrightarrow \Lambda$ and $\mathbf{e}: \Lambda \longrightarrow \Omega$, the natural isomorphism $\gamma^{\mathbf{d}, \mathbf{e}}$ from the composite $\mathbf{e}_\diamond^{\text{pd}} \circ \mathbf{d}_\diamond^{\text{pd}}$ to $(\mathbf{e} \circ \mathbf{d})_\diamond^{\text{pd}}$ is that which is defined, for every S -sorted*

$\mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\mathfrak{p}\mathfrak{d}(\Sigma))$, we have the configuration

$$\begin{array}{ccc}
 (\mathbf{A}, X) & \xrightarrow{(f, P)} & (\mathbf{B}, Y) \\
 \swarrow & & \searrow \\
 (\mathbf{A}, X^\dagger) & \xrightarrow{(f, \beta_\Sigma(P))} & (\mathbf{B}, Y^\dagger) \qquad (\alpha_\Sigma(\mathbf{A}), X) \xrightarrow{(f^\natural, P)} (\alpha_\Sigma(\mathbf{B}), Y)
 \end{array}$$

$$\begin{array}{ccccc}
 & & & & A_X^\natural \\
 & & & & \swarrow \quad \searrow \\
 & & & & P^{\alpha_\Sigma(\mathbf{A})} \\
 & & & & \downarrow \\
 & & & & A_Y^\natural \\
 & & & & \swarrow \quad \searrow \\
 & & & & P^{\alpha_\Sigma(\mathbf{B})} \\
 & & & & \downarrow \\
 & & & & A_Y^\natural \\
 & & & & \swarrow \quad \searrow \\
 & & & & P^{\alpha_\Sigma(\mathbf{B})} \\
 & & & & \downarrow \\
 & & & & A_Y^\natural
 \end{array}$$

and $(\theta_{X,A}^\natural)_{(\mathbf{A},X) \in \mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\mathfrak{p}\mathfrak{d}(\Sigma))}$ is a natural isomorphism. \square

Remark. The functors $\alpha_\Sigma = (\tilde{\mathbf{id}}_{\mathfrak{p}\mathfrak{d}(\Sigma)})_{\mathfrak{p}\mathfrak{d}}^*$ from $\mathbf{Alg}(\Sigma)$ to $\mathbf{Alg}(\mathfrak{p}\mathfrak{d}(\Sigma))$ are the components of a natural transformation α from \mathbf{Alg} into $\mathbf{Alg} \circ \mathfrak{p}\mathfrak{d}^{\text{op}}$, both from \mathbf{Sig}^{op} to \mathbf{Cat} , as in the following diagram

$$\begin{array}{ccc}
 \mathbf{Sig}^{\text{op}} & \xrightarrow{\mathfrak{p}\mathfrak{d}^{\text{op}}} & \mathbf{Sig}^{\text{op}} \\
 \searrow \text{Alg} & & \swarrow \text{Alg} \\
 & \mathbf{Cat} & \\
 & \alpha \cong &
 \end{array}$$

In its turn, the functors $\beta_\Sigma = (\tilde{\mathbf{id}}_{\mathfrak{p}\mathfrak{d}(\Sigma)})_{\mathfrak{d}}^{\mathfrak{p}\mathfrak{d}}$ from $\mathbf{Ter}(\mathfrak{p}\mathfrak{d}(\Sigma))$ to $\mathbf{Ter}(\Sigma)$ are the components of a natural transformation β from $\mathbf{Ter} \circ \mathfrak{p}\mathfrak{d}$ into \mathbf{Ter} , both from \mathbf{Sig} to \mathbf{Cat} , as in the following diagram

$$\begin{array}{ccc}
 \mathbf{Sig} & \xrightarrow{\mathfrak{p}\mathfrak{d}} & \mathbf{Sig} \\
 \searrow \text{Ter} & & \swarrow \text{Ter} \\
 & \mathbf{Cat} & \\
 & \beta \cong &
 \end{array}$$

Besides, if for a polyderivor $\mathbf{d}: \Sigma \longrightarrow \Lambda$ we denote by $\tilde{\mathbf{d}}: \Sigma \longrightarrow \mathfrak{p}\mathfrak{d}(\Lambda)$ the signature morphism associated to \mathbf{d} , by the isomorphism between $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ and $\mathbf{Kl}(\mathbb{T}_{\mathfrak{p}\mathfrak{d}})$ stated in Proposition 52, then we have that

- (1) $\mathbf{d}_{\mathfrak{p}\mathfrak{d}}^* = \tilde{\mathbf{d}}^* \circ \alpha_\Lambda$, and
- (2) $\mathbf{d}_{\mathfrak{d}}^{\mathfrak{p}\mathfrak{d}} = \beta_\Lambda \circ \tilde{\mathbf{d}}_{\mathfrak{d}}$.

Therefore the morphism mappings of the pseudo-functors $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$ and $\mathbf{Ter}_{\mathfrak{p}\mathfrak{d}}$ are definable through the natural transformations α and β , respectively.

In the next proposition we construct a pseudo-functor $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}(\cdot) \times \mathbf{Ter}_{\mathfrak{p}\mathfrak{d}}(\cdot)$ from the product category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}^{\text{op}} \times \mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to \mathbf{Cat} (obtained from the pseudo-functor $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$ and

the pseudo-functor $\text{Ter}_{\text{p}\partial}$), and prove that the family $\text{Tr} = (\text{Tr}^\Sigma)_{\Sigma \in \mathbf{Sig}_{\text{p}\partial}}$, together with the family $\theta = (\theta^{\mathbf{d}})_{\mathbf{d} \in \text{Mor}(\mathbf{Sig}_{\text{p}\partial})}$ is a pseudo-extranatural transformation from the pseudo-functor $\text{Alg}_{\text{p}\partial}(\cdot) \times \text{Ter}_{\text{p}\partial}(\cdot)$ to the functor $\text{K}_{\mathbf{Set}}$ (from $\mathbf{Sig}_{\text{p}\partial}^{\text{op}} \times \mathbf{Sig}_{\text{p}\partial}$ to \mathbf{Cat}) that is constantly \mathbf{Set} .

Proposition 62. *There exists a pseudo-functor $\text{Alg}_{\text{p}\partial}(\cdot) \times \text{Ter}_{\text{p}\partial}(\cdot)$ from the category $\mathbf{Sig}_{\text{p}\partial}^{\text{op}} \times \mathbf{Sig}_{\text{p}\partial}$ to the 2-category \mathbf{Cat} , obtained from the pseudo-functors $\text{Alg}_{\text{p}\partial}$ and $\text{Ter}_{\text{p}\partial}$, which sends a pair of signatures (Σ, Λ) to the category $\mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Lambda)$, and a pair of signature morphisms (\mathbf{d}, \mathbf{e}) from (Σ, Λ) to (Σ', Λ') in $\mathbf{Sig}_{\text{p}\partial}^{\text{op}} \times \mathbf{Sig}_{\text{p}\partial}$ to the functor $\mathbf{d}_{\text{p}\partial}^* \times \mathbf{d}_{\text{p}\partial}^{\text{p}\partial}$ from $\mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Lambda)$ to $\mathbf{Alg}(\Sigma') \times \mathbf{Ter}(\Lambda')$.*

Furthermore, the family of functors $\text{Tr} = (\text{Tr}^\Sigma)_{\Sigma \in \mathbf{Sig}_{\text{p}\partial}}$, together with the family $\theta = (\theta^{\mathbf{d}})_{\mathbf{d} \in \text{Mor}(\mathbf{Sig}_{\text{p}\partial})}$, with $\theta_{\mathbf{A}, X}^{\mathbf{d}} = \theta_{X, A}^{\dagger \mathbf{d}}$, is a pseudo-extranatural transformation from the pseudo-functor $\text{Alg}_{\text{p}\partial}(\cdot) \times \text{Ter}_{\text{p}\partial}(\cdot)$ to the functor $\text{K}_{\mathbf{Set}}$ from $\mathbf{Sig}_{\text{p}\partial}^{\text{op}} \times \mathbf{Sig}_{\text{p}\partial}$ to \mathbf{Cat} that is constantly \mathbf{Set} .

Proof. We restrict ourselves to prove that, for every polyderivator $\mathbf{d}: \Sigma \longrightarrow \Lambda$, the following diagram iso-commutes.

$$\begin{array}{ccc}
 \mathbf{Alg}(\Lambda) \times \mathbf{Ter}(\Sigma) & \xrightarrow{\mathbf{d}_{\text{p}\partial}^* \times \text{Id}} & \mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Sigma) \\
 \text{Id} \times \mathbf{d}_{\text{p}\partial}^{\text{p}\partial} \downarrow & & \downarrow \text{Tr}^\Sigma \\
 \mathbf{Alg}(\Lambda) \times \mathbf{Ter}(\Lambda) & \xrightarrow{\text{Tr}^\Lambda} & \mathbf{Set}
 \end{array}$$

But in the following diagram, where, we recall, $\tilde{\mathbf{d}}: \Sigma \longrightarrow \text{p}\partial(\Lambda)$ is the signature morphism associated to the polyderivator $\mathbf{d}: \Sigma \longrightarrow \Lambda$, by Proposition 52,

$$\begin{array}{ccc}
 \mathbf{Alg}(\Lambda) \times \mathbf{Ter}(\Sigma) & \xrightarrow{\mathbf{d}_{\text{p}\partial}^* \times \text{Id}} & \mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Sigma) \\
 \downarrow \text{Id} \times \mathbf{d}_{\text{p}\partial}^{\text{p}\partial} & \searrow \alpha_\Lambda \times \text{Id} & \uparrow \tilde{\mathbf{d}}^* \times \text{Id} \\
 & & \mathbf{Alg}(\text{p}\partial(\Lambda)) \times \mathbf{Ter}(\Sigma) \\
 & \searrow \text{Id} \times \tilde{\mathbf{d}}_{\text{p}\partial} & \downarrow \text{Id} \times \tilde{\mathbf{d}}_{\text{p}\partial} \quad (2) \\
 & & \mathbf{Alg}(\text{p}\partial(\Lambda)) \times \mathbf{Ter}(\text{p}\partial(\Lambda)) \\
 \downarrow \text{Id} \times \mathbf{d}_{\text{p}\partial}^{\text{p}\partial} & \searrow \alpha_\Lambda \times \text{Id} & \downarrow \text{Tr}^\Sigma \\
 \mathbf{Alg}(\Lambda) \times \mathbf{Ter}(\text{p}\partial(\Lambda)) & \xrightarrow{\alpha_\Lambda \times \text{Id}} & \mathbf{Alg}(\text{p}\partial(\Lambda)) \times \mathbf{Ter}(\text{p}\partial(\Lambda)) \\
 \downarrow \text{Id} \times \beta_\Lambda & \searrow \text{Id} \times \beta_\Lambda & \downarrow \text{Tr}^{\text{p}\partial(\Lambda)} \\
 \mathbf{Alg}(\Lambda) \times \mathbf{Ter}(\Lambda) & \xrightarrow{\text{Tr}^\Lambda} & \mathbf{Set}
 \end{array}$$

we have that the bottom trapezoid (1) iso-commutes by Lemma 8, the right-hand trapezoid (2) iso-commutes because Tr is a pseudo-extranatural transformation and the remaining subdiagrams commute by the definitions of the involved entities. \square

Corollary 17. *The quadruple $\mathfrak{Im}_{\text{p}\partial} = (\mathbf{Sig}_{\text{p}\partial}, \text{Alg}_{\text{p}\partial}, \text{Ter}_{\text{p}\partial}, (\text{Tr}, \theta))$ is an institution on \mathbf{Set} , the so-called many-sorted term institution of Fujiwara, or, to abbreviate, the term institution of Fujiwara.*

Remark. Since every standard signature morphism can be identified to a polyderivator, the term institution is canonically embedded into the term institution of Fujiwara.

We close this section by constructing, for the category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$, the many-sorted equational institution $\mathfrak{L}\mathfrak{E}\mathfrak{q}_{\mathfrak{p}\mathfrak{d}}$. To do it we define a pseudo-functor $\mathbf{LEq}_{\mathfrak{p}\mathfrak{d}}$ on $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$.

Definition 33. We denote by $\mathbf{LEq}_{\mathfrak{p}\mathfrak{d}}$ the pseudo-functor from $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to $\mathbf{Cat}_{\mathcal{V}}$ given by the following data

- (1) The object mapping of $\mathbf{LEq}_{\mathfrak{p}\mathfrak{d}}$ is that which sends a signature Σ to the discrete category $\mathbf{LEq}(\Sigma)$ canonically associated to the set

$$\bigcup_{X, Y \in \mathcal{U}} (\text{Hom}(Y, T_{\Sigma}(X)))^2 \times \{(X, Y)\}$$

of labelled Σ -equations, i.e., the set of all pairs $((P, Q), (X, Y))$ with (P, Q) a Σ -equation of type (X, Y) , for some $X, Y \in \mathcal{U}$.

- (2) The morphism mapping of $\mathbf{LEq}_{\mathfrak{p}\mathfrak{d}}$ is that which sends a polyderivator \mathbf{d} from Σ to Λ to the functor $\mathbf{LEq}_{\mathfrak{p}\mathfrak{d}}(\mathbf{d})$ from $\mathbf{LEq}(\Sigma)$ to $\mathbf{LEq}(\Lambda)$ which assigns to the labelled equation $((P, Q), (X, Y))$ in $\mathbf{LEq}(\Sigma)$ the labelled equation

$$\mathbf{LEq}_{\mathfrak{p}\mathfrak{d}}(\mathbf{d})((P, Q), (X, Y)) = ((\mathbf{d}_{\circ}^{\mathfrak{p}\mathfrak{d}}(P), \mathbf{d}_{\circ}^{\mathfrak{p}\mathfrak{d}}(Q)), (\coprod_{\varphi}^{\dagger} X, \coprod_{\varphi}^{\dagger} Y))$$

in $\mathbf{LEq}(\Lambda)$.

Corollary 18. *The quadruple $\mathfrak{L}\mathfrak{E}\mathfrak{q}_{\mathfrak{p}\mathfrak{d}} = (\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}, \text{Alg}_{\mathfrak{p}\mathfrak{d}}, \mathbf{LEq}_{\mathfrak{p}\mathfrak{d}}, (\models, \theta))$ is an institution on $\mathbf{2}$, the so-called many-sorted equational institution of Fujiwara, or, simply, the equational institution of Fujiwara.*

6. TRANSFORMATIONS OF FUJIWARA.

Continuing the work begun in [21], Fujiwara defines in [22] an equivalence relation, the *conjugation*, on the set of families of basic mapping-formulas from a given single-sorted signature into a like one, relative to a set of equations for the target signature. But to properly define such an equivalence relation it is necessary to define beforehand, for two families of basic mapping-formulas (Φ, P) , with $\Phi = \{\varphi_{\mu} \mid \mu \in p\}$, and (Ψ, Q) , with $\Psi = \{\psi_{\nu} \mid \nu \in q\}$, from a single-sorted signature Σ into a like one Λ , the concept of transformation from (Φ, P) to (Ψ, Q) , relative to a set of equations for the target signature. One such transformation will be a mapping $L: \Psi \longrightarrow T_{\Lambda}(\Phi)$ subject to satisfy a certain compatibility condition that involves P, Q , and the given set of equations.

Next we proceed to give a short, but precise, description of the transformations between families of basic mapping-formulas and of the derived equivalence relation of conjugation as stated by Fujiwara in [22]. Let L be a mapping from Ψ to $T_{\Lambda}(\Phi)$, i.e., a family $(L_{\nu})_{\nu \in q}$ of terms for Λ with variables in Φ . Then, for every $\nu \in q$, we get

$$L_{\nu}^{\mathbf{T}_{\Lambda}(\Phi \times \downarrow v_n)}: T_{\Lambda}(\Phi \times \downarrow v_n)^{\Phi} \longrightarrow T_{\Lambda}(\Phi \times \downarrow v_n),$$

the term operation on $\mathbf{T}_{\Lambda}(\Phi \times \downarrow v_n)$ determined by L_{ν} . This follows from the fact that, for every $\nu \in q$, L_{ν} belongs to $T_{\Lambda}(\Phi)$, and because, by the universal property of the free Λ -algebra on Φ , we have the Λ -homomorphism $(\text{pr}_{\mu})_{\mu \in p}^{\sharp}$ from $\mathbf{T}_{\Lambda}(\Phi)$ to

$\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^\Phi}$, as seen in the following diagram

$$\begin{array}{ccc}
 & & \Psi \\
 & & \downarrow L \\
 \Phi & \xrightarrow{\eta_\Phi} & \mathbf{T}_\Lambda(\Phi) \\
 & \searrow (\text{pr}_\mu)_{\mu \in p} & \downarrow (\text{pr}_\mu)_{\mu \in p}^\# (= \text{Tr}^{\Phi, \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)}) \\
 & & \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^\Phi}
 \end{array}$$

where, for every $\mu \in p$, pr_μ is the μ -th projection from $\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^\Phi$ to $\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)$.

But, for every $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, we have that $(P_{\varphi_\mu, \sigma}^n)_{\mu \in p} \in \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^\Phi$, hence, for every $\nu \in q$, the term operation $L_\nu^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)}$ is such that

$$L_\nu^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)} \begin{cases} \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^\Phi & \longrightarrow \mathbf{T}_\Lambda(\Phi \times \downarrow v_n) \\ (P_{\varphi_\mu, \sigma}^n)_{\mu \in p} & \longmapsto L_\nu^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)}((P_{\varphi_\mu, \sigma}^n)_{\mu \in p}) \end{cases}$$

i.e., it sends, in particular, the given family of terms $(P_{\varphi_\mu, \sigma}^n)_{\mu \in p}$ in $\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)$ to the term $L_\nu^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)}((P_{\varphi_\mu, \sigma}^n)_{\mu \in p})$ in $\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)$.

On the other hand, given $n \in \mathbb{N}$ and $i \in n$ we have the following commutative diagram

$$\begin{array}{ccc}
 \Phi & \xrightarrow{\eta_\Phi} & \mathbf{T}_\Lambda(\Phi) \\
 \langle \text{id}_\Phi, \kappa_{v_i} \rangle \downarrow & & \downarrow \langle \text{id}_\Phi, \kappa_{v_i} \rangle^\circledast \\
 \Phi \times \downarrow v_n & \xrightarrow{\eta_{\Phi \times \downarrow v_n}} & \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)
 \end{array}$$

where $\langle \text{id}_\Phi, \kappa_{v_i} \rangle$ is the mapping which sends φ_μ in Φ to (φ_μ, v_i) in $\Phi \times \downarrow v_n$, and $\langle \text{id}_\Phi, \kappa_{v_i} \rangle^\circledast = (\eta_{\Phi \times \downarrow v_n} \circ \langle \text{id}_\Phi, \kappa_{v_i} \rangle)^\#$ is the underlying mapping of the value of the functor \mathbf{T}_Λ in $\langle \text{id}_\Phi, \kappa_{v_i} \rangle$. Then, from the family of mappings $(\langle \text{id}_\Phi, \kappa_{v_i} \rangle^\circledast)_{i \in n}$, and since we dispose of the mapping $L: \Psi \longrightarrow \mathbf{T}_\Lambda(\Phi)$, we get the family of terms

$$((\langle \text{id}_\Phi, \kappa_{v_i} \rangle^\circledast(L_\nu))_{\nu \in q})_{i \in n} \in (\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^\Psi)^n,$$

or, because of the isomorphism $(\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^\Psi)^n \cong \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^{\Psi \times \downarrow v_n}$, the matrix

$$\begin{pmatrix} \langle \text{id}_\Phi, \kappa_{v_0} \rangle^\circledast(L_0) & \cdots & \langle \text{id}_\Phi, \kappa_{v_{n-1}} \rangle^\circledast(L_0) \\ \vdots & \ddots & \vdots \\ \langle \text{id}_\Phi, \kappa_{v_0} \rangle^\circledast(L_{q-1}) & \cdots & \langle \text{id}_\Phi, \kappa_{v_{n-1}} \rangle^\circledast(L_{q-1}) \end{pmatrix}$$

or, equivalently, by the exchange law (which, in this case, says that, for $i \in n$, $\nu \in q$, valuations $\eta_{\Phi \times \downarrow v_n} \circ \langle \text{id}_\Phi, \kappa_{v_i} \rangle: \Phi \longrightarrow \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)$, and terms $L_\nu \in \mathbf{T}_\Lambda(\Phi)$, we have that $\langle \text{id}_\Phi, \kappa_{v_i} \rangle^\circledast(L_\nu) = L_\nu^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)}(\varphi_\mu(v_i) \mid \mu \in p)$), the matrix

$$\begin{pmatrix} L_0^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)}(\varphi_\mu(v_0) \mid \mu \in p) & \cdots & L_0^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)}(\varphi_\mu(v_{n-1}) \mid \mu \in p) \\ \vdots & \ddots & \vdots \\ L_{q-1}^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)}(\varphi_\mu(v_0) \mid \mu \in p) & \cdots & L_{q-1}^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)}(\varphi_\mu(v_{n-1}) \mid \mu \in p) \end{pmatrix}$$

with entries in $\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)$.

Besides, by hypothesis, for every $n \in \mathbb{N}$ and $\sigma \in \Sigma_n$, we have the family of terms

$$(Q_{\psi_\nu, \sigma}^n)_{\nu \in q}: \Psi \longrightarrow \mathbf{T}_\Lambda(\Psi \times \downarrow v_n).$$

Therefore, for every $\nu \in q$, we get

$$Q_{\psi\nu,\sigma}^{n,\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)}: \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^{\Psi \times \downarrow v_n} \longrightarrow \mathbf{T}_\Lambda(\Phi \times \downarrow v_n),$$

the term operation on $\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)$ determined by $Q_{\psi\nu,\sigma}^n$. This follows from the fact that, for every $\nu \in q$, $Q_{\psi\nu,\sigma}^n$ is in $\mathbf{T}_\Lambda(\Psi \times \downarrow v_n)$, and because, by the universal property of the free Λ -algebra on $\Psi \times \downarrow v_n$, we have the Λ -homomorphism $(\text{pr}_{\nu,i})_{(\nu,i) \in q \times n}^\sharp$ from $\mathbf{T}_\Lambda(\Psi \times \downarrow v_n)$ to $\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^{\Psi \times \downarrow v_n}}$, as seen in the following diagram

$$\begin{array}{ccc} & & \Psi \\ & & \downarrow \\ & & (Q_{\psi\nu,\sigma}^n)_{\nu \in q} \\ & & \downarrow \\ \Psi \times \downarrow v_n & \xrightarrow{\eta_{\Psi \times \downarrow v_n}} & \mathbf{T}_\Lambda(\Psi \times \downarrow v_n) \\ & \searrow (\text{pr}_{\nu,i})_{(\nu,i) \in q \times n} & \downarrow (\text{pr}_{\nu,i})_{(\nu,i) \in q \times n}^\sharp (= \text{Tr}^{\Psi \times \downarrow v_n, \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)}) \\ & & \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^{\Psi \times \downarrow v_n}} \end{array}$$

where, for every $(\nu, i) \in q \times n$, $\text{pr}_{\nu,i}$ is the (ν, i) -th projection from $\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^{\Psi \times \downarrow v_n}$ to $\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)$.

Let $\langle Q_{\psi\nu,\sigma}^{n,\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)} \rangle_{\nu \in q}$ be the unique Λ -homomorphism from $\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^{\Psi \times \downarrow v_n}$ to $\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^\Psi$ such that, for every $\nu \in q$, the following diagram commutes

$$\begin{array}{ccc} & \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^{\Psi \times \downarrow v_n} & \\ & \downarrow \langle Q_{\psi\nu,\sigma}^{n,\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)} \rangle_{\nu \in q} & \searrow Q_{\psi\nu,\sigma}^{n,\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)} \\ & \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^\Psi & \xrightarrow{\text{pr}_\nu} \mathbf{T}_\Lambda(\Phi \times \downarrow v_n) \end{array}$$

Then, since

$$((\langle \text{id}_\Phi, \kappa_{v_i} \rangle^\otimes(L_\nu))_{\nu \in q})_{i \in n} \in (\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^\Psi)^n \cong \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^{\Psi \times \downarrow v_n},$$

the family $\left(Q_{\psi\nu,\sigma}^{n,\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)} \left(\begin{array}{ccc} \langle \text{id}_\Phi, \kappa_{v_0} \rangle^\otimes(L_0) & \cdots & \langle \text{id}_\Phi, \kappa_{v_{n-1}} \rangle^\otimes(L_0) \\ \vdots & \ddots & \vdots \\ \langle \text{id}_\Phi, \kappa_{v_0} \rangle^\otimes(L_{q-1}) & \cdots & \langle \text{id}_\Phi, \kappa_{v_{n-1}} \rangle^\otimes(L_{q-1}) \end{array} \right) \right)_{\nu \in q}$ belongs to $\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^\Psi$.

After stating these technical preliminaries, and assuming as given a set of equations \mathcal{H} for Λ , finally we are ready to provide the definition proposed by Fujiwara in [22] of the concept of transformation from (Φ, P) to (Ψ, Q) . It will be a mapping $L: \Psi \longrightarrow \mathbf{T}_\Lambda(\Phi)^q$, or what is equivalent an element of $\mathbf{T}_\Lambda(\Phi)^q$, such that, for every $\nu \in q$, the equation

$$Q_{\psi\nu,\sigma}^{n,\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)} \left(\begin{array}{ccc} \langle \text{id}_\Phi, \kappa_{v_0} \rangle^\otimes(L_0) & \cdots & \langle \text{id}_\Phi, \kappa_{v_{n-1}} \rangle^\otimes(L_0) \\ \vdots & \ddots & \vdots \\ \langle \text{id}_\Phi, \kappa_{v_0} \rangle^\otimes(L_{q-1}) & \cdots & \langle \text{id}_\Phi, \kappa_{v_{n-1}} \rangle^\otimes(L_{q-1}) \end{array} \right) = L_\nu^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)}((P_{\varphi_\mu,\sigma}^n)_{\mu \in p})$$

holds (not necessarily strictly but only) modulus the set of equations \mathcal{H} .

From this, for two single-sorted signatures Σ, Λ , and a set of equations \mathcal{H} for Λ , Fujiwara defines in [22] an equivalence relation on the set of families of basic mapping-formulas from Σ to Λ (that he calls *B_W-conjugacy*, with W identified to Λ and B_W to \mathcal{H}), by saying that two families of basic mapping-formulas (Φ, P) and (Ψ, Q) from Σ to Λ are equivalent, relative to the set of equations \mathcal{H} for

Λ , if there exists a transformation $L: \Psi \longrightarrow T_\Lambda(\Phi)$ from (Φ, P) to (Ψ, Q) such that, for every (Λ, \mathcal{H}) -algebra \mathbf{B} the mapping $\langle L_\nu^\mathbf{B} \rangle_{\nu \in q}$, from $P(\mathbf{B})$ to $Q(\mathbf{B})$ is an isomorphism. To this we add that Fujiwara in [22] considers the vertical composition of transformations between families of basic mapping-formulas, in order to prove the transitivity, but he does not consider the horizontal composition of transformations between families of basic mapping-formulas between single-sorted signatures (no wonder, in his time there was not any explicit notion of 2-category).

Example. For the derivors $d: \Sigma \longrightarrow \Lambda$ and $e: \Lambda \longrightarrow \Sigma$ of Higman and B.H. Neumann defined in the preceding section, where

- (1) $\Sigma_0 = \{1\}$, $\Sigma_2 = \{/ \}$ and $\Sigma_n = \emptyset$, if $n \neq 0, 2$,
- (2) $\Lambda_0 = \{1\}$, $\Lambda_1 = \{-^1\}$, $\Lambda_2 = \{\cdot\}$ and $\Lambda_n = \emptyset$, if $n \neq 0, 1, 2$,
- (3) d defines $/$ in terms of \cdot and $^{-1}$, and
- (4) e defines \cdot and $^{-1}$ in terms of $/$,

if we take as set \mathcal{H} of defining axioms, relative to the signature Σ , that system given by the following (unusual) group axioms

- (1) $v_0 / (((v_0/v_0)/v_1)/v_2) / (((v_0/v_0)/v_0)/v_2) = v_1$, and
- (2) $v_0/1 = v_0$,

then, taking as L the term in $T_\Sigma(1)$ whose term realization on any given Σ -algebra is, essentially, the corresponding identity mapping, we get a transformation from the endoderivor $e \circ d$ at Σ to the identity endoderivor at Σ because the equation

$$v_0 / (1 / (1/v_1)) = v_0/v_1$$

holds, modulus \mathcal{H} .

There is also a similar transformation from the endoderivor $d \circ e$ at Λ to the identity endoderivor at Λ .

Example. For the derivors $d: \Sigma \longrightarrow \Lambda$ and $e: \Lambda \longrightarrow \Sigma$ of M. H. Stone defined in the preceding section where

- (1) $\Sigma_0 = \{0, 1\}$, $\Sigma_1 = \{'\}$, $\Sigma_2 = \{\wedge, \vee\}$ and $\Sigma_n = \emptyset$, if $n \neq 0, 1, 2$,
- (2) $\Lambda_0 = \{0, 1\}$, $\Lambda_1 = \{-\}$, $\Lambda_2 = \{\cdot, +\}$ and $\Lambda_n = \emptyset$, if $n \neq 0, 1, 2$,
- (3) d defines the ‘‘Boolean algebra’’ operations in terms of the ‘‘Boolean ring’’ operations, and
- (4) e defines the ‘‘Boolean ring’’ operations in terms of the ‘‘Boolean algebra’’ operations,

if we take as set \mathcal{H} of defining axioms, relative to the signature Λ , that system given by the usual axioms for a Boolean ring, then, taking as L the term in $T_\Lambda(1)$ whose term realization on any given Λ -algebra is, essentially, the corresponding identity mapping, we get a transformation from the endoderivor $d \circ e$ at Λ to the identity endoderivor at Λ because the equations

- (1) $(v_0 \cdot (1 + v_1)) + ((1 + v_0) \cdot v_1) + ((v_0 \cdot (1 + v_1)) \cdot ((1 + v_0) \cdot v_1)) = v_0 + v_1$,
- (2) $v_0 \cdot v_1 = v_0 \cdot v_1$, and
- (3) $-v_0 = 1 + v_0$,

hold, modulus \mathcal{H} .

Example. In [22], pp. 260–268, Fujiwara provides examples of transformations for families of basic mapping-formulas of derivation-type when the operations and equations are those corresponding to commutative linear algebras over a given field of characteristic zero.

Having just sketched the theory of Fujiwara about the transformations between families of basic mapping-formulas, what we ultimately try to do is to define (once we have at our disposal a convenient notion of morphism from a specification into

a like one, but for polyderivors), for two morphisms \mathbf{d} , \mathbf{e} from a specification (Σ, \mathcal{E}) into a like one (Λ, \mathcal{H}) a concept of transformation from \mathbf{d} to \mathbf{e} , as well as a vertical and a horizontal composition for these transformations, and all that in such a way that specifications, morphisms and transformations constitute a 2-category. But to succeed in doing it we should begin, as we do in this section, by defining a structure of 2-category on the category \mathbf{Sig}_{pd} , of signatures and polyderivors, through the concept of transformation between polyderivors.

The transformations between polyderivors that we define below are a generalization (up to the many-sorted case) of the above concept of transformation between families of basic mapping-formulas, due to Fujiwara. But, because the polyderivors are, simply, morphisms from a signature into a like one, and not morphisms between specifications (where there are involved equations), they will satisfy, for every formal operation, a strict equation, instead of an equation modulus a set of equations for the target signature. However, after we define in the last section the adequate morphisms between specifications (through the polyderivors between the underlying signatures of the specifications), we will get, also in that section, the full generalization of the theory of Fujiwara in [22], as announced above.

In this section we also prove that the transformations between polyderivors determine natural transformations between the functors associated to the polyderivors, that allow us to lift the pseudo-functors Alg_{pd} and Ter_{pd} up to 2-functors, and hence to get, by applying a construction of Ehresmann-Grothendieck to Alg_{pd} , a 2-category \mathbf{Alg}_{pd} . Besides, we prove that the transformations between polyderivors are also compatible with the realization of the terms in the algebras and we characterize this through the concept of pseudo-extranatural transformation between pseudo-functors on 2-categories. From this we get that the relation between terms and algebras is an example of 2-institution.

In order to define and investigate the transformations between polyderivors it will be shown to be convenient to make use of some derived operations in the Bénabou algebras of terms for the different signatures, concretely of those in the following

Definition 34. Let S be a set of sorts.

- (1) For every $\bar{w} \in S^{**}$ and $\alpha \in |\bar{w}|$, let $\pi_\alpha^{\bar{w}}$ be the derived operation of type $\lambda \longrightarrow (\lambda \bar{w}, \bar{w}_\alpha)$ defined as

$$\langle \pi_{\sum_{\beta \in \alpha} p_\beta}^{\lambda \bar{w}}, \dots, \pi_{\sum_{\beta \in \alpha+1} p_{\beta-1}}^{\lambda \bar{w}} \rangle_{\lambda \bar{w}, \bar{w}_\alpha},$$

where \bar{w} is of the form

$$((\cdot, \dots, \cdot), \dots, \overbrace{(\cdot, \dots, \cdot)}^{\bar{w}_\alpha}, \dots, (\cdot, \dots, \cdot)),$$

and, for every $\alpha \in |\bar{w}|$, $p_\alpha = |\bar{w}_\alpha|$.

- (2) For every $u \in S^*$ and $\bar{w} \in S^{**}$, let $\langle \rangle_{u, \bar{w}}$ be the derived operation of type $((u, \bar{w}_0), \dots, (u, \bar{w}_{|\bar{w}|-1})) \longrightarrow (u, \lambda \bar{w})$ defined as

$$\langle P_0, \dots, P_{|\bar{w}|-1} \rangle_{u, \bar{w}} = \langle \pi_0^{\bar{w}_0} \circ P_0, \dots, \pi_{|\bar{w}_0|-1}^{\bar{w}_0} \circ P_0, \dots, \pi_0^{\bar{w}_{|\bar{w}|-1}} \circ P_{|\bar{w}|-1}, \dots, \pi_{|\bar{w}_{|\bar{w}|-1}|-1}^{\bar{w}_{|\bar{w}|-1}} \circ P_{|\bar{w}|-1} \rangle_{u, \lambda \bar{w}}.$$

- (3) For every $n \in \mathbb{N}$, and $\bar{u}, \bar{w} \in S^{*n}$, let $\lambda_{\bar{u}, \bar{w}}$ be the derived operation of type $((\bar{u}_0, \bar{w}_0), \dots, (\bar{u}_{n-1}, \bar{w}_{n-1})) \longrightarrow (\lambda \bar{u}, \lambda \bar{w})$ defined as

$$\lambda_{\bar{u}, \bar{w}}(P_0, \dots, P_{n-1}) = \langle P_0 \circ \pi_0^{\bar{u}}, \dots, P_{n-1} \circ \pi_{n-1}^{\bar{u}} \rangle_{\lambda \bar{u}, \bar{w}}.$$

From now on, to simplify the notation, we will omit some subscripts in the expressions. Moreover, for the operations of the form $\lambda_{\bar{u}, \bar{w}}$ we adopt the infix

notation, and we will write $P_0 \wedge \cdots \wedge P_{n-1}$ instead of $\wedge_{\bar{u}, \bar{w}}(P_0, \dots, P_{n-1})$, the type, in its turn, will be $\bar{u}_0 \wedge \cdots \wedge \bar{u}_{n-1} \longrightarrow \bar{w}_0 \wedge \cdots \wedge \bar{w}_{n-1}$.

For the algebras of terms $\mathbf{BTer}_S(\Sigma)$, the operations $\wedge_{\bar{u}, \bar{w}}$ are, essentially, the result of gathering into a family the corresponding terms, relabelling adequately the variables.

Recalling that the Bénabou algebras are, up to isomorphism, the finitary many-sorted algebraic theories of Bénabou (see Proposition 44), from now on, we will represent the composition of terms diagrammatically, and the equality of two cotermininal paths composed of terms by asserting the commutativity of the appropriate diagram.

Definition 35. Let \mathbf{d} and \mathbf{e} be polyderivors from Σ to Λ . A *transformation from \mathbf{d} to \mathbf{e}* is a choice function ξ for $(\mathbf{BTer}_T(\Lambda))_{\varphi(s), \psi(s)}_{s \in S} = (\mathbf{T}_\Lambda(\downarrow\varphi(s))_{\psi(s)})_{s \in S}$, i.e., an element of $\prod_{s \in S} \mathbf{T}_\Lambda(\downarrow\varphi(s))_{\psi(s)}$, such that, for every operation $\sigma: w \longrightarrow s$, the following diagram commutes

$$\begin{array}{ccc} 1 & \xrightarrow{\langle \xi_s, d(\sigma) \rangle} & \mathbf{T}_\Lambda(\downarrow\varphi(s))_{\psi(s)} \times \mathbf{T}_\Lambda(\downarrow\varphi^\sharp(w))_{\varphi(s)} \\ \langle e(\sigma), \xi_w \rangle \downarrow & & \downarrow \circ \\ \mathbf{T}_\Lambda(\downarrow\psi^\sharp(w))_{\psi(s)} \times \mathbf{T}_\Lambda(\downarrow\varphi^\sharp(w))_{\psi^\sharp(w)} & \xrightarrow{\circ} & \mathbf{T}_\Lambda(\downarrow\varphi^\sharp(w))_{\psi(s)} \end{array}$$

or more briefly, such that

$$\xi_s \circ d(\sigma) = e(\sigma) \circ \xi_w,$$

where ξ_w is $\xi_{w_0} \wedge \cdots \wedge \xi_{w_{|w|-1}}$. We agree upon writing $\xi: \mathbf{d} \rightsquigarrow \mathbf{e}$ to denote the fact that ξ is a transformation from \mathbf{d} to \mathbf{e} .

Therefore for a transformation $\xi = (\xi_s)_{s \in S}$ from \mathbf{d} to \mathbf{e} we have, in particular, that, for every $s \in S$, $\xi_s \in \mathbf{T}_\Lambda(\downarrow\varphi(s))_{\psi(s)}$, i.e., that

$$\xi_s = ((\xi_s)_0, \dots, (\xi_s)_{|\psi(s)|-1})$$

is a tuple of length $|\psi(s)|$ such that, for every $i \in |\psi(s)|$, $(\xi_s)_i$ is a term for Λ of type $(\downarrow\varphi(s), \psi(s)_i)$.

Henceforth, we agree to represent the commutativity condition for a transformation $\xi: \mathbf{d} \rightsquigarrow \mathbf{e}$ between polyderivors also by the following diagram

$$\begin{array}{ccc} \varphi^\sharp(w) & \xrightarrow{d(\sigma)} & \varphi(s) \\ \xi_w \downarrow & & \downarrow \xi_s \\ \psi^\sharp(w) & \xrightarrow{e(\sigma)} & \psi(s) \end{array}$$

where, we recall, ξ_w arises as

$$\begin{array}{ccc} \varphi(w_0) \wedge \cdots \wedge \varphi(w_{|w|-1}) & \xrightarrow{\pi_i^{\varphi^*(w)}} & \varphi(w_i) \\ \xi_w \downarrow & & \downarrow \xi_{w_i} \\ \psi(w_0) \wedge \cdots \wedge \psi(w_{|w|-1}) & \xrightarrow{\pi_i^{\psi^*(w)}} & \psi(w_i) \end{array}$$

Example. Let Σ be a signature, $p, q \in \mathbb{N}$, and $\mathbf{d} = (\varphi, d)$, $\mathbf{e} = (\psi, e)$ two polyderivators from Σ into itself, such that

- (1) $\varphi: S \longrightarrow S^*$ is the mapping which sends $s \in S$ to the word $\lambda_{\mu \in p}(s)$ and,
- (2) For $(w, s) \in S^* \times S$, $d_{w,s}$ is the mapping from $\Sigma_{w,s}$ to $\mathbf{T}_\Sigma(\downarrow \varphi^\sharp(w))_s^p$ which sends $\sigma \in \Sigma_{w,s}$ to

$$(\sigma(v_0^{w_0}, v_p^{w_1}, \dots, v_{(|w|-1)p}^{w_{|w|-1}}), \dots, \sigma(v_{p-1}^{w_0}, v_{2p-1}^{w_1}, \dots, v_{|w|p-1}^{w_{|w|-1}})),$$

and

- (1) $\psi: S \longrightarrow S^*$ is the mapping which sends $s \in S$ to the word $\lambda_{\nu \in q}(s)$ and,
- (2) For $(w, s) \in S^* \times S$, $e_{w,s}$ is the mapping from $\Sigma_{w,s}$ to $\mathbf{T}_\Sigma(\downarrow \psi^\sharp(w))_s^q$ which sends $\sigma \in \Sigma_{w,s}$ to

$$(\sigma(v_0^{w_0}, v_q^{w_1}, \dots, v_{(|w|-1)q}^{w_{|w|-1}}), \dots, \sigma(v_{q-1}^{w_0}, v_{2q-1}^{w_1}, \dots, v_{|w|q-1}^{w_{|w|-1}})).$$

Then, for an arbitrary, but fixed, mapping $f = (f(\nu))_{\nu \in q}$ from the natural number q to the natural number p , taking as ξ the element of $\prod_{s \in S} \mathbf{T}_\Lambda(\downarrow \varphi(s))_s^q$ defined, for every $s \in S$, as

$$\xi_s = (v_{f(0)}^s, \dots, v_{f(q-1)}^s),$$

where, to simplify the notation, we have identified the variables in $\downarrow \varphi(s)$ with their images in $\mathbf{T}_\Sigma(\downarrow \varphi(s))$ under $\eta_{\downarrow \varphi(s)}$, we have that ξ is a transformation from \mathbf{d} , \mathbf{e} . We point out that the working out of all the details of this example, even if a little troublesome, helps to grasp the functioning of the polyderivators and the transformations between them.

For more examples of transformations between polyderivators we refer to the last section of this paper.

Remark. In the just stated example, due to the intended meaning of ξ as a ms-mapping from the direct p -power to the direct q -power of some Σ -algebra, the mappings of the type $f: q \longrightarrow p$ act by selecting the coordinates for going from the first direct power to the second direct power.

Example. Let (Φ, P) , with $\Phi = \{\varphi_\mu \mid \mu \in p\}$, and (Ψ, Q) , with $\Psi = \{\psi_\nu \mid \nu \in q\}$, be two families of basic mapping-formulas from the single-sorted signature Σ to the single-sorted signature Λ , and $L \in \mathbf{T}_\Lambda(\Phi)^q$. Then L is a transformation from the polyderivator associated to (Φ, P) to the polyderivator associated to (Ψ, Q) iff, for every $n \in \mathbb{N}$ and every $\sigma \in \Sigma_n$, the following diagram commutes

$$\begin{array}{ccc} 1 & \xrightarrow{\langle L, d_{n,0}(\sigma) \rangle} & \mathbf{T}_\Lambda(\Phi)^q \times \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^p \\ \langle e_{n,0}(\sigma), L_n \rangle \downarrow & & \downarrow \circ \\ \mathbf{T}_\Lambda(\Psi \times \downarrow v_n)^q \times \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^{\Psi \times \downarrow v_n} & \xrightarrow{\circ} & \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)^q \end{array}$$

where

- (1) $d_{n,0}(\sigma) = (P_{\varphi_0, \sigma}^n, \dots, P_{\varphi_{p-1}, \sigma}^n)$,
- (2) $e_{n,0}(\sigma) = (Q_{\psi_0, \sigma}^n, \dots, Q_{\psi_{q-1}, \sigma}^n)$, and
- (3) $L_n = \begin{pmatrix} \langle \text{id}_\Phi, \kappa_{v_0} \rangle^{\otimes (L_0)} & \cdots & \langle \text{id}_\Phi, \kappa_{v_{n-1}} \rangle^{\otimes (L_0)} \\ \vdots & \ddots & \vdots \\ \langle \text{id}_\Phi, \kappa_{v_0} \rangle^{\otimes (L_{q-1})} & \cdots & \langle \text{id}_\Phi, \kappa_{v_{n-1}} \rangle^{\otimes (L_{q-1})} \end{pmatrix}$.

Observe that in this case the right-down path in the diagram is the family

$$(L_\nu^{\mathbf{T}_\Lambda(\Phi \times \downarrow v_n)}((P_{\varphi_\mu, \sigma}^n)_{\mu \in p}))_{\nu \in q},$$

while the down-right path is the family

$$\left(Q_{\psi_\nu, \sigma}^{n, \mathbf{T}_\Lambda(\Phi \times \downarrow v_n)} \left(\begin{array}{ccc} \langle \text{id}_\Phi, \kappa_{v_0} \rangle^{\otimes} (L_0) & \cdots & \langle \text{id}_\Phi, \kappa_{v_{n-1}} \rangle^{\otimes} (L_0) \\ \vdots & \ddots & \vdots \\ \langle \text{id}_\Phi, \kappa_{v_0} \rangle^{\otimes} (L_{q-1}) & \cdots & \langle \text{id}_\Phi, \kappa_{v_{n-1}} \rangle^{\otimes} (L_{q-1}) \end{array} \right) \right)_{\nu \in q}.$$

Therefore, without having a set of equations \mathcal{H} for the target single-sorted signature Λ , any transformation of Fujiwara from a family of basic mapping-formulas into a like one is an example of transformation between the polyderivators associated to the families of basic mapping-formulas.

The commutativity condition in the above definition of transformation from a polyderivator into a like one can be extended up to the terms, as proved in the following

Proposition 63. *Let \mathbf{d} and \mathbf{e} be polyderivators from Σ to Λ and $\xi: \mathbf{d} \rightsquigarrow \mathbf{e}$ a transformation. Then, for every term $P: u \multimap w$ in $\text{BTer}_S(\Sigma)$, $\xi_w \circ d^\sharp(P) = e^\sharp(P) \circ \xi_u$, i.e., the following diagram commutes*

$$\begin{array}{ccc} \varphi^\sharp(u) & \xrightarrow{d^\sharp(P)} & \varphi^\sharp(w) \\ \xi_u \downarrow & & \downarrow \xi_w \\ \psi^\sharp(u) & \xrightarrow{e^\sharp(P)} & \psi^\sharp(w) \end{array}$$

Proof. By algebraic induction in the Bénabou algebra $\text{BTer}_S(\Sigma)$. The basis of the induction holds because it means that ξ is a transformation.

For the operations π_i^w , we have that $d^\sharp(\pi_i^w) = \pi_i^{\varphi^*(w)}$, $e^\sharp(\pi_i^w) = \pi_i^{\psi^*(w)}$, and

$$\xi_{w_i} \circ \pi_i^{\varphi^*(w)} = \pi_i^{\psi^*(w)} \circ \xi_w,$$

i.e., the following diagram commutes

$$\begin{array}{ccc} \varphi^\sharp(w) & \xrightarrow{\pi_i^{\varphi^*(w)}} & \varphi(w_i) \\ \xi_w \downarrow & & \downarrow \xi_{w_i} \\ \psi^\sharp(u) & \xrightarrow{\pi_i^{\psi^*(w)}} & \psi(w_i) \end{array}$$

For the operations $\langle \rangle_{u,w}$, we have that

$$\xi_w \circ d^\sharp(\langle P_0, \dots, P_{|w|-1} \rangle_{u,w}) = e^\sharp(\langle P_0, \dots, P_{|w|-1} \rangle_{u,w}) \circ \xi_u,$$

i.e., that the following diagram commutes

$$\begin{array}{ccc} \varphi^\sharp(u) & \xrightarrow{d^\sharp(\langle P_0, \dots, P_{|w|-1} \rangle_{u,w})} & \varphi^\sharp(w) \\ \xi_u \downarrow & & \downarrow \xi_w \\ \psi^\sharp(u) & \xrightarrow{e^\sharp(\langle P_0, \dots, P_{|w|-1} \rangle_{u,w})} & \psi^\sharp(w) \end{array}$$

because

$$\begin{aligned}
\xi_w \circ d^\#(\langle P_0, \dots, P_{|w|-1} \rangle_{u,w}) &= \langle d^\#(P_0), \dots, d^\#(P_{|w|-1}) \rangle_{\varphi^\#(u), \varphi^*(w)} \\
&= \langle (\xi_{w_i} \circ d^\#(P_i) \mid i \in |w|) \rangle_{\varphi^\#(u), \varphi^*(w)} \\
&= \langle e^\#(P_0) \circ \xi_u, \dots, e^\#(P_{|w|-1}) \circ \xi_u \rangle_{\varphi^\#(u), \varphi^*(w)} \\
&= \langle e^\#(P_0), \dots, e^\#(P_{|w|-1}) \rangle_{\varphi^\#(u), \varphi^*(w)} \circ \xi_u \\
&= e^\#(\langle P_0, \dots, P_{|w|-1} \rangle_{u,w}) \circ \xi_u.
\end{aligned}$$

Finally, for the operations $\circ_{u,x,w}$, it is obvious that

$$\xi_x \circ d^\#(Q) \circ d^\#(P) = e^\#(Q) \circ e^\#(P) \circ \xi_u,$$

i.e., that the following diagram commutes

$$\begin{array}{ccccc}
\varphi^\#(u) & \xrightarrow{d^\#(P)} & \varphi^\#(w) & \xrightarrow{d^\#(Q)} & \varphi^\#(x) \\
\xi_u \downarrow & & \downarrow \xi_w & & \downarrow \xi_x \\
\psi^\#(u) & \xrightarrow{e^\#(P)} & \psi^\#(w) & \xrightarrow{e^\#(Q)} & \psi^\#(x)
\end{array}$$

□

What we want now is to endow the category \mathbf{Sig}_{pd} of signatures and polyderivors with a structure of 2-category. For this we provide in the following proposition the definitions of the horizontal and vertical composition of the transformations between polyderivors, prove the law of Godement, and define the identity transformations at the polyderivors.

Proposition 64. *The signatures together with the polyderivors and the transformations between the polyderivors have a structure of 2-category, denoted as \mathbf{Sig}_{pd} .*

Proof. Definition of the vertical composition. Given the configuration

$$\begin{array}{ccc}
& \mathbf{d} & \\
& \curvearrowright \xi & \\
\Sigma & \xrightarrow{\quad} \mathbf{e} & \xrightarrow{\quad} \Lambda \\
& \curvearrowleft \chi & \\
& \mathbf{h} &
\end{array}$$

the vertical composition of ξ and χ , denoted by $\chi \circ \xi$ and defined as

$$\chi \circ \xi = (\chi_s \circ \xi_s)_{s \in S},$$

is a transformation from \mathbf{d} to \mathbf{h} , because, for every $\sigma: w \longrightarrow s$, the following diagram commutes

$$\begin{array}{ccc}
\varphi^\#(w) & \xrightarrow{d(\sigma)} & \varphi(s) \\
\xi_w \downarrow & & \downarrow \xi_s \\
\psi^\#(w) & \xrightarrow{e(\sigma)} & \psi(s) \\
\chi_w \downarrow & & \downarrow \chi_s \\
\gamma^\#(w) & \xrightarrow{h(\sigma)} & \gamma(s)
\end{array}$$

Definition of the horizontal composition. Given the configuration

$$\begin{array}{ccccc} & \mathbf{d} & & \mathbf{h} & \\ \Sigma & \curvearrowright & & \curvearrowleft & \Omega \\ & \downarrow \xi & \Lambda & \downarrow \chi & \\ & \mathbf{e} & & \mathbf{i} & \end{array}$$

the horizontal composition of ξ and χ , denoted by $\chi * \xi$ and defined as

$$\chi * \xi = (\chi_{\psi(s)} \circ h^\sharp(\xi_s))_{s \in S},$$

or, equivalently, as $(i^\sharp(\xi_s) \circ \chi_{\varphi(s)})_{s \in S}$, is a transformation from $\mathbf{h} \circ \mathbf{d}$ to $\mathbf{i} \circ \mathbf{e}$. We have to prove that $\chi * \xi$ is a transformation from $(\gamma^\sharp \circ \varphi, h^\sharp_{\varphi^\sharp \times \varphi^\sharp} \circ d)$ to $(\nu^\sharp \circ \psi, i^\sharp_{\psi^\sharp \times \psi^\sharp} \circ e)$, i.e., that, for every $\sigma: w \rightarrow s$, we have that

$$(\chi * \xi)_s \circ h^\sharp(d(\sigma)) = i^\sharp(e(\sigma)) \circ (\chi * \xi)_w.$$

But this happens since ξ, χ are transformations and h^\sharp, i^\sharp morphisms, i.e., because the following diagram commutes

$$\begin{array}{ccccc} & \gamma^\sharp(\varphi^\sharp(w)) & \xrightarrow{h^\sharp(d(\sigma))} & & \gamma^\sharp(\varphi(s)) \\ & \swarrow h^\sharp(\xi_w) & \downarrow \chi_{\varphi^\sharp(w)} & \swarrow h^\sharp(\xi_s) & \downarrow \chi_{\varphi(s)} \\ \gamma^\sharp(\psi^\sharp(w)) & \xrightarrow{h^\sharp(e(\sigma))} & \gamma^\sharp(\psi(s)) & & \\ \downarrow \chi_{\psi^\sharp(w)} & & \downarrow \chi_{\psi(s)} & & \downarrow \chi_{\psi(s)} \\ & \nu^\sharp(\varphi^\sharp(w)) & \xrightarrow{i^\sharp(d(\sigma))} & & \nu^\sharp(\varphi(s)) \\ & \swarrow i^\sharp(\xi_w) & \downarrow i^\sharp(e(\sigma)) & \swarrow i^\sharp(\xi_s) & \\ \nu^\sharp(\psi^\sharp(w)) & \xrightarrow{i^\sharp(e(\sigma))} & \nu^\sharp(\psi(s)) & & \end{array}$$

Law of Godement. Given the configuration

$$\begin{array}{ccccc} & \mathbf{d}_0 & & \mathbf{e}_0 & \\ \Sigma & \curvearrowright & & \curvearrowleft & \Omega \\ & \downarrow \xi & \Lambda & \downarrow \xi' & \\ & \mathbf{d}_1 & & \mathbf{e}_1 & \\ & \downarrow \chi & & \downarrow \chi' & \\ & \mathbf{d}_2 & & \mathbf{e}_2 & \end{array}$$

we have that

$$(\chi' * \chi) \circ (\xi' * \xi) = (\chi' \circ \xi') * (\chi \circ \xi).$$

This is so since the following diagram commutes

$$\begin{array}{ccccc}
& & e_0^\sharp(\chi_s \circ \xi_s) & & \\
& & \curvearrowright & & \\
& \psi_0^\sharp(\varphi_0(s)) & \xrightarrow{e_0^\sharp(\xi_s)} & \psi_0^\sharp(\varphi_1(s)) & \xrightarrow{e_0^\sharp(\chi_s)} & \psi_0^\sharp(\varphi_2(s)) & \\
& \downarrow \xi'_{\varphi_0(s)} & & \downarrow \xi'_{\varphi_1(s)} & & \downarrow \xi'_{\varphi_2(s)} & \\
(\chi' \circ \xi')_{\varphi_0(s)} & \psi_1^\sharp(\varphi_0(s)) & \xrightarrow{e_1^\sharp(\xi_s)} & \psi_1^\sharp(\varphi_1(s)) & \xrightarrow{e_1^\sharp(\chi_s)} & \psi_1^\sharp(\varphi_2(s)) & (\chi' \circ \xi')_{\varphi_2(s)} \\
& \downarrow \chi'_{\varphi_0(s)} & & \downarrow \chi'_{\varphi_1(s)} & & \downarrow \chi'_{\varphi_2(s)} & \\
& \psi_2^\sharp(\varphi_0(s)) & \xrightarrow{e_2^\sharp(\xi_s)} & \psi_2^\sharp(\varphi_1(s)) & \xrightarrow{e_2^\sharp(\chi_s)} & \psi_2^\sharp(\varphi_2(s)) & \\
& & \curvearrowleft & & & e_2^\sharp(\chi_s \circ \xi_s) &
\end{array}$$

Identities. Finally, given polyderivator $\mathbf{d}: \Sigma \longrightarrow \mathbf{A}$ and $\mathbf{e}: \mathbf{A} \longrightarrow \mathbf{Q}$ it is obvious that

- (1) The S -family $(\langle \pi_0^{\varphi(s)}, \dots, \pi_{|\varphi(s)|-1}^{\varphi(s)} \rangle_{\varphi(s), \varphi(s)})_{s \in S}$, denoted by $\text{id}_{\mathbf{d}}$, is the identity transformation at \mathbf{d} , and that
- (2) $\text{id}_{\mathbf{e}} * \text{id}_{\mathbf{d}} = \text{id}_{\mathbf{e} \circ \mathbf{d}}$. □

Our next goal is to prove that the transformations between polyderivors from a signature into a like one, determine natural transformations between the functors between the categories of algebras associated to the signatures. To accomplish this we begin by proving that every transformation ξ from a polyderivator \mathbf{d} to another one \mathbf{e} , both from a signature Σ to a signature \mathbf{A} , determines, for a given \mathbf{A} -algebra \mathbf{B} , a Σ -homomorphism $\xi^{\mathbf{B}}$ from $\mathbf{d}_{\text{pd}}^*(\mathbf{B})$ to $\mathbf{e}_{\text{pd}}^*(\mathbf{B})$.

Proposition 65. *Let \mathbf{d} and \mathbf{e} be polyderivors from Σ to \mathbf{A} , $\xi: \mathbf{d} \rightsquigarrow \mathbf{e}$ a transformation in \mathbf{Sig}_{pd} , and, for a \mathbf{A} -algebra $\mathbf{B} = (B, G)$, let $\xi^{\mathbf{B}}$ be the S -sorted mapping $(\xi_s^{\mathbf{B}})_{s \in S}$ from B_φ to B_ψ , where, for every $s \in S$, we have that*

$$\xi_s^{\mathbf{B}} = G_{\varphi(s), \psi(s)}^\sharp(\xi_s): B_{\varphi(s)} \longrightarrow B_{\psi(s)}.$$

Then $\xi^{\mathbf{B}}$ is a Σ -homomorphism from $\mathbf{d}_{\text{pd}}^(\mathbf{B})$ to $\mathbf{e}_{\text{pd}}^*(\mathbf{B})$.*

Proof. For every operation $\sigma: w \longrightarrow s$, in Σ , we have to prove that $G_\sigma^{\mathbf{e}} \circ \xi_w^{\mathbf{B}} = \xi_s^{\mathbf{B}} \circ G_\sigma^{\mathbf{d}}$, and for this it is enough to prove that every face, up to at most the frontal one, in the following diagram commutes

$$\begin{array}{ccccc}
& & B_{\varphi^\sharp(w)} & \xrightarrow{(G_{\varphi^\sharp \times \varphi^\sharp}^\sharp \circ d)_{w,s}(\sigma)} & B_{\varphi^\sharp(s)} & \xrightarrow{(\iota_{\varphi^\sharp(s)}^B)^{-1}} & B_{\varphi_s} \\
& \nearrow \iota_{\varphi^\sharp(w)}^B & \downarrow & \xrightarrow{G_\sigma^{\mathbf{d}}} & \downarrow & \searrow & \downarrow \\
B_{\varphi_w} & \xrightarrow{(\xi_w^{\mathbf{B}})^{\mathbf{B}}} & B_{\varphi^\sharp(w)} & \xrightarrow{(G_{\varphi^\sharp \times \varphi^\sharp}^\sharp \circ d)_{w,s}(\sigma)} & B_{\varphi^\sharp(s)} & \xrightarrow{(\iota_{\varphi^\sharp(s)}^B)^{-1}} & B_{\varphi_s} \\
& \downarrow (\xi_w^{\mathbf{B}})^{\mathbf{B}} & \downarrow & \xrightarrow{G_\sigma^{\mathbf{e}}} & \downarrow & \searrow & \downarrow \\
& \nearrow \iota_{\psi^\sharp(w)}^B & B_{\psi^\sharp(w)} & \xrightarrow{(G_{\psi^\sharp \times \psi^\sharp}^\sharp \circ e)_{w,s}(\sigma)} & B_{\psi^\sharp(s)} & \xrightarrow{(\iota_{\psi^\sharp(s)}^B)^{-1}} & B_{\psi_s} \\
& \downarrow (\xi_w^{\mathbf{B}})^{\mathbf{B}} & \downarrow & \xrightarrow{G_\sigma^{\mathbf{e}}} & \downarrow & \searrow & \downarrow \\
B_{\psi_w} & \xrightarrow{(\xi_w^{\mathbf{B}})^{\mathbf{B}}} & B_{\psi^\sharp(w)} & \xrightarrow{(G_{\psi^\sharp \times \psi^\sharp}^\sharp \circ e)_{w,s}(\sigma)} & B_{\psi^\sharp(s)} & \xrightarrow{(\iota_{\psi^\sharp(s)}^B)^{-1}} & B_{\psi_s}
\end{array}$$

from which it also follows, necessarily, that the frontal face also commutes.

The top and bottom faces commute by definition. The back face commutes because, being ξ a transformation from \mathbf{d} to \mathbf{e} , from the commutativity of the following diagram

$$\begin{array}{ccc} \varphi^\sharp(w) & \xrightarrow{d(\sigma)} & \varphi(s) \\ \xi_w \downarrow & & \downarrow \xi_s \\ \psi^\sharp(w) & \xrightarrow{e(\sigma)} & \psi(s) \end{array}$$

it follows that

$$\begin{aligned} (\xi_s)^\mathbf{B} \circ (G_{\varphi^\sharp \times \varphi^\sharp}^\sharp \circ d)_{w,s}(\sigma) &= G_{\varphi(s), \psi(s)}^\sharp(\xi_s) \circ G_{\varphi^\sharp(w), \varphi(s)}^\sharp(d_{w,s}(\sigma)) \\ &= G_{\varphi^\sharp(w), \psi(s)}^\sharp(\xi_s \circ d_{w,s}(\sigma)) \\ &= G_{\varphi^\sharp(w), \psi(s)}^\sharp(e_{w,s}(\sigma) \circ \xi_w) \\ &= G_{\psi^\sharp(w), \psi(s)}^\sharp(e_{w,s}(\sigma)) \circ G_{\varphi^\sharp(w), \psi^\sharp(w)}^\sharp(\xi_w) \\ &= (G_{\psi^\sharp \times \psi^\sharp}^\sharp \circ e)_{w,s}(\sigma) \circ (\xi_w)^\mathbf{B}. \end{aligned}$$

Relative to the lateral faces, let us verify, e.g., that the left one commutes. For this it is enough to prove that

$$(\xi_w)^\mathbf{B} = \iota_{\psi^\sharp(w)}^B \circ \xi_w^\mathbf{B} \circ (\iota_{\varphi^\sharp(w)}^B)^{-1}.$$

But we have that

$$\begin{aligned} (\xi_w)^\mathbf{B} &= G_{\varphi^\sharp(w), \psi^\sharp(w)}^\sharp(\xi_{w_0} \wedge \cdots \wedge \xi_{w_{|w|-1}}) \\ &= G_{\varphi(w_0), \psi(w_0)}^\sharp(\xi_{w_0}) \wedge \cdots \wedge G_{\varphi(w_{|w|-1}), \psi(w_{|w|-1})}^\sharp(\xi_{w_{|w|-1}}) \\ &= \xi_{w_0}^\mathbf{B} \wedge \cdots \wedge \xi_{w_{|w|-1}}^\mathbf{B} \\ &= \langle \xi_{w_0}^\mathbf{B} \circ \text{pr}_{(0)}^{B_{\varphi^\sharp(w)}}, \dots, \xi_{w_{|w|-1}}^\mathbf{B} \circ \text{pr}_{(|w|-1)}^{B_{\varphi^\sharp(w)}} \rangle \end{aligned}$$

hence it is enough to prove that, for every $i \in |w|$, we have that

$$\text{pr}_{(i)}^{B_{\psi^\sharp(w)}} \circ \iota_{\psi^\sharp(w)}^B \circ \xi_w^\mathbf{B} \circ (\iota_{\varphi^\sharp(w)}^B)^{-1} = \xi_{w_i}^\mathbf{B} \circ \text{pr}_{(i)}^{B_{\varphi^\sharp(w)}},$$

but this follows from the commutativity of the following diagram

$$\begin{array}{ccccc} & & B_{\varphi^\sharp(w)} & & \\ & \swarrow (\iota_{\varphi^\sharp(w)}^B)^{-1} & & \searrow \text{pr}_{(i)}^{B_{\varphi^\sharp(w)}} & \\ B_{\varphi_w} & & & & B_{\varphi(w_i)} \\ & \xrightarrow{\text{pr}_i^{B_{\varphi_w}}} & & & \\ (\xi_w^\mathbf{B}) \downarrow & & & & \downarrow \xi_{w_i}^\mathbf{B} \\ B_{\psi_w} & & & & B_{\psi(w_i)} \\ & \xrightarrow{\text{pr}_i^{B_{\psi_w}}} & & & \\ & \swarrow \iota_{\psi^\sharp(w)}^B & & \searrow \text{pr}_{(i)}^{B_{\psi^\sharp(w)}} & \\ & & B_{\psi^\sharp(w)} & & \end{array}$$

□

After having proved, for two polyderivors \mathbf{d} and \mathbf{e} from Σ to Λ , that every transformation ξ from \mathbf{d} to \mathbf{e} , induces, for every Λ -algebra \mathbf{B} , a Σ -homomorphism $\xi^{\mathbf{B}}$ from $\mathbf{d}_{\mathfrak{p}\mathfrak{d}}^*(\mathbf{B})$ to $\mathbf{e}_{\mathfrak{p}\mathfrak{d}}^*(\mathbf{B})$, we prove in the following proposition the naturalness of the involved process.

Proposition 66. *Let $\xi: \mathbf{d} \rightsquigarrow \mathbf{e}$ be a transformation with \mathbf{d} and \mathbf{e} polyderivors from Σ to Λ . Then the family $(\xi^{\mathbf{B}})_{\mathbf{B} \in \mathbf{Alg}(\Lambda)}$, denoted by $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}(\xi)$, is a natural transformation from the functor $\mathbf{d}_{\mathfrak{p}\mathfrak{d}}^*$ to the functor $\mathbf{e}_{\mathfrak{p}\mathfrak{d}}^*$, both from $\mathbf{Alg}(\Lambda)$ to $\mathbf{Alg}(\Sigma)$.*

Proof. We have to prove that, for every Λ -algebras $\mathbf{B} = (B, G)$, $\mathbf{C} = (C, H)$ and morphism $f: \mathbf{B} \rightarrow \mathbf{C}$ in $\mathbf{Alg}(\Lambda)$, the following diagram commutes

$$\begin{array}{ccc} (B_{\varphi}, G^{\mathbf{d}}) & \xrightarrow{\xi^{\mathbf{B}}} & (B_{\psi}, G^{\mathbf{e}}) \\ f_{\varphi} \downarrow & & \downarrow f_{\psi} \\ (C_{\varphi}, H^{\mathbf{d}}) & \xrightarrow{\xi^{\mathbf{C}}} & (C_{\psi}, H^{\mathbf{e}}) \end{array}$$

But this is immediate because, for every $s \in S$, we have that $\xi_s^{\mathbf{B}}$ and $\xi_s^{\mathbf{C}}$ are the realizations of the term ξ_s in the respective algebras, hence the following diagram commutes

$$\begin{array}{ccc} B_{\varphi(s)} & \xrightarrow{\xi_s^{\mathbf{B}}} & B_{\psi(s)} \\ f_{\varphi(s)} \downarrow & & \downarrow f_{\psi(s)} \\ C_{\varphi(s)} & \xrightarrow{\xi_s^{\mathbf{C}}} & C_{\psi(s)} \end{array}$$

□

Once stated that the transformations between polyderivors from a signature into a like one, induce natural transformations among the functors between the categories of algebras associated to the signatures, we can properly lift the pseudo-functor $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}: \mathbf{Sig}_{\mathfrak{p}\mathfrak{d}} \rightarrow \mathbf{Cat}$ up to the 2-cells in the 2-category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$.

Proposition 67. *There exists a pseudo-functor $\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}$, contravariant in the morphisms and covariant in the 2-cells, from the 2-category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to the 2-category \mathbf{Cat} given schematically by the following data*

$$\begin{array}{ccc} \mathbf{Sig}_{\mathfrak{p}\mathfrak{d}} & \xrightarrow{\mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}} & \mathbf{Cat} \\ \\ \begin{array}{ccc} \Sigma & & \\ \mathbf{d} \downarrow \xi \rightsquigarrow \mathbf{e} & & \\ \Lambda & & \end{array} & \mapsto & \begin{array}{ccc} \mathbf{Alg}(\Sigma) & & \\ \mathbf{d}_{\mathfrak{p}\mathfrak{d}}^* \downarrow \mathbf{Alg}_{\mathfrak{p}\mathfrak{d}}(\xi) \Rightarrow \mathbf{e}_{\mathfrak{p}\mathfrak{d}}^* & & \\ \mathbf{Alg}(\Lambda) & & \end{array} \end{array}$$

together with the accompanying natural isomorphisms $\gamma^{\mathbf{d}, \mathbf{e}}$ and ν^{Σ} , as defined in Proposition 55.

Proof. It follows from the fact that the natural isomorphisms of the pseudo-functor are compatible with the structure of 2-category of $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$. □

On the basis of this last proposition we can lift the category $\mathbf{Alg}_{p\partial}$ up to a 2-category as in the following

Definition 36. We denote by $\mathbf{Alg}_{p\partial} = \iint^{\mathbf{Sig}_{p\partial}} \mathbf{Alg}_{p\partial}$ the 2-category which has

- (1) As objects (0-cells) the pairs (Σ, \mathbf{A}) , where Σ is a signature and \mathbf{A} a Σ -algebra,
- (2) As morphisms (1-cells) from (Σ, \mathbf{A}) to (Λ, \mathbf{B}) the pairs (\mathbf{d}, f) , where \mathbf{d} is a polyderivator from Σ to Λ and f a Σ -homomorphism from \mathbf{A} to $\mathbf{d}_{p\partial}^*(\mathbf{B})$, and
- (3) As 2-cells from (\mathbf{d}, f) to (\mathbf{e}, g) , where (\mathbf{d}, f) and (\mathbf{e}, g) are morphisms from (Σ, \mathbf{A}) to (Λ, \mathbf{B}) , the 2-cells $\xi: \Sigma \rightsquigarrow \Lambda$ in $\mathbf{Sig}_{p\partial}$ such that the following diagram commutes

$$\begin{array}{ccc}
& & \mathbf{d}_{p\partial}^*(\mathbf{B}) \\
& \nearrow f & \downarrow \xi^{\mathbf{B}} \\
\mathbf{A} & & \mathbf{e}_{p\partial}^*(\mathbf{B}) \\
& \searrow g &
\end{array}$$

Relative to the above 2-category we point out that for the following configuration of 2-cells

$$\begin{array}{ccccc}
(\Sigma, \mathbf{A}) & \xrightarrow{(\mathbf{d}, f)} & (\Lambda, \mathbf{B}) & \xrightarrow{(\mathbf{h}, p)} & (\Omega, \mathbf{C}) \\
& \searrow \xi & & \searrow \chi & \\
& (\mathbf{e}, g) & \rightarrow & (\mathbf{i}, q) & \rightarrow
\end{array}$$

we have the commutative diagram

$$\begin{array}{ccccccc}
& & & & & & (\mathbf{h} \circ \mathbf{d})_{p\partial}^*(\mathbf{C}) \\
& & & & \mathbf{d}_{p\partial}^*(p) & \rightarrow & \downarrow \xi^{\mathbf{h} \circ \mathbf{d}} \\
& & \mathbf{d}_{p\partial}^*(\mathbf{B}) & \xrightarrow{\mathbf{d}_{p\partial}^*(q)} & (\mathbf{i} \circ \mathbf{d})_{p\partial}^*(\mathbf{C}) & \xleftarrow{\mathbf{d}_{p\partial}^*(\chi^{\mathbf{C}})} & \\
& \nearrow f & \downarrow \xi^{\mathbf{B}} & & \downarrow \xi^{\mathbf{i} \circ \mathbf{d}} & & \downarrow \xi^{\mathbf{h} \circ \mathbf{d}} \\
\mathbf{A} & & \mathbf{e}_{p\partial}^*(\mathbf{B}) & \xrightarrow{\mathbf{e}_{p\partial}^*(p)} & (\mathbf{h} \circ \mathbf{e})_{p\partial}^*(\mathbf{C}) & \xleftarrow{\mathbf{e}_{p\partial}^*(\chi^{\mathbf{C}})} & \\
& \searrow g & \downarrow \xi^{\mathbf{B}} & & \downarrow \xi^{\mathbf{i} \circ \mathbf{e}} & & \downarrow \xi^{\mathbf{h} \circ \mathbf{e}} \\
& & & & (\mathbf{i} \circ \mathbf{e})_{p\partial}^*(\mathbf{C}) & &
\end{array}$$

As was the case above for algebras and transformations, our goal now is to prove that the transformations between polyderivators from a signature into a like one, also determine natural transformations between the functors between the categories of terms associated to the signatures. To accomplish this we begin by proving that every transformation ξ from a polyderivator \mathbf{d} to another one \mathbf{e} , both from a signature Σ to a signature Λ , determines, for a given S -sorted set X , a morphism ξ_X , in the category $\mathbf{Ter}(\Lambda)$, from $\prod_{\varphi}^{\dagger} X$ to $\prod_{\psi}^{\dagger} X$.

Proposition 68. Let \mathbf{d} and \mathbf{e} be polyderivators from Σ to Λ , $\xi: \mathbf{d} \rightsquigarrow \mathbf{e}$ a transformation in $\mathbf{Sig}_{p\partial}$, and, for an S -sorted set X , let $\xi_X: \prod_{\psi}^{\dagger} X \longrightarrow \mathbf{T}_{\Lambda}(\prod_{\varphi}^{\dagger} X)$ be the T -sorted mapping defined, for $t \in T$ and $(x, s, \psi(s), i) \in (\prod_{\psi}^{\dagger} X)_t$, as follows

$$(\xi_X)_t(x, s, \psi(s), i) = (\xi_s)_i(v_j^{\varphi(s)_j} / (x, s, \varphi(s), j) \mid j \in |\varphi(s)|).$$

Then the mapping ξ_X is a morphism, in the category $\mathbf{Ter}(\Lambda)$, from $\prod_{\varphi}^{\dagger} X$ to $\prod_{\psi}^{\dagger} X$.

Proof. The definition of $\xi_X: \coprod_{\varphi}^{\dagger} X \longrightarrow T_{\Lambda}(\coprod_{\varphi}^{\dagger} X)$ is sound since, for every $j \in |\varphi(s)|$, we have that $(x, s, \varphi(s), j) \in (\coprod_{\varphi}^{\dagger} X)_{\varphi(s)_j}$ and $(\xi_s)_i \in T_{\Lambda}(\varphi(s))_{\psi(s)_i}$, hence $(\xi_X)_t(x, s, \varphi(s), i)$ is a term for Λ of type $\psi(s)_i = t$. \square

After having proved, for two polyderivors \mathbf{d} and \mathbf{e} from Σ to Λ , that every transformation ξ from \mathbf{d} to \mathbf{e} , induces, for every S -sorted set X , a morphism ξ_X from $\coprod_{\varphi}^{\dagger} X$ to $\coprod_{\psi}^{\dagger} X$, we prove in the following proposition that they are the components of a natural transformation.

Proposition 69. *Let $\xi: \mathbf{d} \rightsquigarrow \mathbf{e}$ be a transformation in $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$, with \mathbf{d}, \mathbf{e} polyderivors from Σ to Λ . Then $\text{Ter}_{\mathfrak{p}\mathfrak{d}}(\xi) = (\xi_X)_{X \in \mathbf{Ter}(\Sigma)}$ is a natural transformation from $\mathbf{d}_{\diamond}^{\mathfrak{p}\mathfrak{d}}$ to $\mathbf{e}_{\diamond}^{\mathfrak{p}\mathfrak{d}}$.*

Proof. Because, for a morphism $P: X \longrightarrow Y$ in $\mathbf{Ter}(\Sigma)$, the following diagram commutes

$$\begin{array}{ccc} \coprod_{\varphi}^{\dagger} X & \xrightarrow{\xi_X} & \coprod_{\psi}^{\dagger} X \\ \mathbf{d}_{\diamond}^{\mathfrak{p}\mathfrak{d}}(P) \downarrow & & \downarrow \mathbf{e}_{\diamond}^{\mathfrak{p}\mathfrak{d}}(P) \\ \coprod_{\varphi}^{\dagger} Y & \xrightarrow{\xi_Y} & \coprod_{\psi}^{\dagger} Y \end{array}$$

\square

Observe that this last proposition is analogous to Proposition 63 but for derived operations with variables in arbitrary many-sorted sets.

Once stated that the transformations between polyderivors from a signature into a like one, induce natural transformations among the functors between the categories of terms associated to the signatures, we can properly lift the pseudo-functor $\text{Ter}_{\mathfrak{p}\mathfrak{d}}: \mathbf{Sig}_{\mathfrak{p}\mathfrak{d}} \longrightarrow \mathbf{Cat}$ up to the 2-cells of the 2-category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$.

Proposition 70. *There exists a pseudo-functor $\text{Ter}_{\mathfrak{p}\mathfrak{d}}$ from the 2-category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to \mathbf{Cat} , covariant in the morphisms and the 2-cells, given schematically by the following data*

$$\begin{array}{ccc} \mathbf{Sig}_{\mathfrak{p}\mathfrak{d}} & \xrightarrow{\text{Ter}_{\mathfrak{p}\mathfrak{d}}} & \mathbf{Cat} \\ \\ \mathbf{d} \begin{array}{c} \curvearrowright \\ \Sigma \\ \curvearrowleft \\ \xi \\ \curvearrowright \\ \Lambda \\ \curvearrowleft \\ \mathbf{e} \end{array} & \longmapsto & \mathbf{d}_{\diamond}^{\mathfrak{p}\mathfrak{d}} \begin{array}{c} \curvearrowright \\ \mathbf{Ter}(\Sigma) \\ \curvearrowleft \\ \text{Ter}_{\mathfrak{p}\mathfrak{d}}(\xi) \\ \curvearrowright \\ \mathbf{e}_{\diamond}^{\mathfrak{p}\mathfrak{d}} \\ \curvearrowleft \\ \mathbf{Ter}(\Lambda) \end{array} \end{array}$$

together with the accompanying natural isomorphisms $\gamma^{\mathbf{d}, \mathbf{e}}$ and ν^{Σ} , as defined in Proposition 61.

Proof. It follows from the fact that the natural isomorphisms of the pseudo-functor are compatible with the structure of 2-category of $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$. \square

Up to this point what we have at our disposal consists, essentially, of the following data:

- (1) The pseudo-functor $\text{Alg}_{\mathfrak{p}\mathfrak{d}}$, contravariant in the polyderivors and covariant in the transformations, from the 2-category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to the 2-category \mathbf{Cat} ,

- (2) The pseudo-functor $\text{Ter}_{\mathbf{pd}}$, covariant in the polyderivors and the transformations, from the 2-category $\mathbf{Sig}_{\mathbf{pd}}$ to the 2-category \mathbf{Cat} ,
- (3) The family of functors $\text{Tr} = (\text{Tr}^{\Sigma})_{\Sigma \in \mathbf{Sig}_{\mathbf{pd}}}$, and
- (4) The family of natural isomorphisms $\theta = (\theta^{\mathbf{d}})_{\mathbf{d} \in \text{Mor}(\mathbf{Sig}_{\mathbf{pd}})}$, and
- (5) Taking $\mathbf{Sig}_{\mathbf{pd}}$ as an ordinary category, the institution $\mathfrak{Im}_{\mathbf{pd}}$ on \mathbf{Set} .

But it happens that $\mathbf{Sig}_{\mathbf{pd}}$ is a 2-category, hence our next goal will be to show that $\mathfrak{Im}_{\mathbf{pd}}$ is not only an institution on \mathbf{Set} but actually a 2-institution on \mathbf{Set} . To attain the just stated goal we should begin by proving that the realization of the terms in the algebras is compatible with the additional structure derived from the 2-cells in $\mathbf{Sig}_{\mathbf{pd}}$, i.e., the transformations between polyderivors.

Lemma 9. *Let $\xi: \mathbf{d} \rightsquigarrow \mathbf{e}$ be a transformation in $\mathbf{Sig}_{\mathbf{pd}}$ from the polydivor \mathbf{d} to the polydivor \mathbf{e} , both from Σ to Λ . Then, for every Λ -algebra \mathbf{B} , and S -sorted set X , the mappings $(\xi^{\mathbf{B}})_X \circ (\theta_{\varphi}^{\dagger \mathbf{d}})_{X,B}$ and $(\theta_{\psi}^{\dagger \mathbf{e}})_{X,B} \circ (\xi_X)^{\mathbf{B}}$ from $B_{\amalg_{\varphi}^{\dagger} X}$ to $(\Delta_{\psi}^{\dagger} B)_X$ are identical, i.e., the following diagram commutes*

$$\begin{array}{ccc}
 B_{\amalg_{\varphi}^{\dagger} X} & \xrightarrow{(\xi_X)^{\mathbf{B}}} & B_{\amalg_{\psi}^{\dagger} X} \\
 (\theta_{\varphi}^{\dagger \mathbf{d}})_{X,B} \downarrow & & \downarrow (\theta_{\psi}^{\dagger \mathbf{e}})_{X,B} \\
 (\Delta_{\varphi}^{\dagger} B)_X & \xrightarrow{(\xi^{\mathbf{B}})_X} & (\Delta_{\psi}^{\dagger} B)_X
 \end{array}$$

Proof. For every $f \in B_{\amalg_{\varphi}^{\dagger} X}$, $(\xi_X)^{\mathbf{B}}(f) \in B_{\amalg_{\psi}^{\dagger} X}$ is the morphism $f^{\#} \circ \xi_X$, where $f^{\#}$ is the extension of f to $\mathbf{T}_{\Lambda}(\amalg_{\varphi}^{\dagger} X)$, obtained as shown in the following diagram

$$\begin{array}{ccccc}
 \amalg_{\varphi}^{\dagger} X & \xrightarrow{\eta_{\amalg_{\varphi}^{\dagger} X}} & \mathbf{T}_{\Lambda}(\amalg_{\varphi}^{\dagger} X) & \xleftarrow{\xi_X} & \amalg_{\psi}^{\dagger} X \\
 & \searrow f & \downarrow f^{\#} & & \swarrow (\xi_X)^{\mathbf{B}}(f) \\
 & & B & &
 \end{array}$$

hence $(\theta_{\psi}^{\dagger \mathbf{e}})_{X,B}((\xi_X)^{\mathbf{B}}(f))$ is a morphism from X to $\Delta_{\psi}^{\dagger} B$. Now, for $s \in S$ and $x \in X_s$, we have that

$$\begin{aligned}
 ((\theta_{\psi}^{\dagger \mathbf{e}})_{X,B}((\xi_X)^{\mathbf{B}}(f)))_s(x) &= ((\theta_{\psi}^{\dagger \mathbf{e}})_{X,B}(f^{\#} \circ \xi_X))_s(x) \\
 &= ((f^{\#} \circ \xi_X)_{\psi(s)_i}(x, s, \psi(s), i) \mid i \in |\psi(s)|) \\
 &= (f_{\psi(s)_i}^{\#}((\xi_X)_{\psi(s)_i}(x, s, \psi(s), i) \mid i \in |\psi(s)|) \\
 &= f_{\psi(s)}^{\#}((\xi_X)_{\psi(s)_i}(x, s, \psi(s), i) \mid i \in |\psi(s)|) \\
 &= f_{\psi(s)}^{\#}((\xi_s)_i(v_j^{\varphi(s)j} / (x, s, \varphi(s), j) \mid j \in |\varphi(s)|_{i \in |\psi(s)|})) \\
 &= f_{\psi(s)}^{\#}(\xi_s^{\mathbf{T}_{\Lambda}(\amalg_{\varphi}^{\dagger} X)}((x, s, \varphi(s), j) \mid j \in |\varphi(s)|)) \\
 &= \xi_s^{\mathbf{B}}(f_{\varphi(s)}^{\#}((x, s, \varphi(s), j) \mid j \in |\varphi(s)|))
 \end{aligned}$$

because $\text{Alg}_{\mathfrak{p}\mathfrak{d}}(\xi)$ is natural and $f^\#$ a morphism

$$\begin{aligned} &= \xi_s^{\mathbf{B}} (f_{\varphi(s)_j}^\# (x, s, \varphi(s), j) \mid j \in |\varphi(s)|) \\ &= \xi_s^{\mathbf{B}} ((\theta_\varphi^{\dagger\mathfrak{h}})_{X,B}(f)_s(x)) \\ &= ((\xi^{\mathbf{B}})_X((\theta_\varphi^{\dagger\mathfrak{h}})_{X,B}(f)))_s(x). \end{aligned}$$

Therefore $(\xi^{\mathbf{B}})_X \circ (\theta_\varphi^{\dagger\mathfrak{h}})_{X,B} = (\theta_\psi^{\dagger\mathfrak{h}})_{X,B} \circ (\xi_X)^{\mathbf{B}}$, as asserted. \square

In the following proposition, which will be the basis to get the many-sorted term 2-institution of Fujiwara on \mathbf{Set} , we construct a pseudo-functor $\text{Alg}_{\mathfrak{p}\mathfrak{d}}(\cdot) \times \text{Ter}_{\mathfrak{p}\mathfrak{d}}(\cdot)$ from the 2-category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}^{\text{op}} \times \mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to the 2-category \mathbf{Cat} (obtained from the pseudo-functors Alg and Ter), and prove that the family $\text{Tr} = (\text{Tr}^\Sigma)_{\Sigma \in \mathbf{Sig}}$, together with the family $\theta = (\theta^{\mathfrak{d}})_{\mathfrak{d} \in \text{Mor}(\mathbf{Sig})}$ is a pseudo-extranatural transformation from the pseudo-functor $\text{Alg}_{\mathfrak{p}\mathfrak{d}}(\cdot) \times \text{Ter}_{\mathfrak{p}\mathfrak{d}}(\cdot)$ to the functor $\text{K}_{\mathbf{Set}}$ from $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}^{\text{op}} \times \mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to \mathbf{Cat} that is constantly \mathbf{Set} .

Proposition 71. *There exists a pseudo-functor $\text{Alg}_{\mathfrak{p}\mathfrak{d}}(\cdot) \times \text{Ter}_{\mathfrak{p}\mathfrak{d}}(\cdot)$ from the 2-category $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}^{\text{op}} \times \mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to the 2-category \mathbf{Cat} , obtained from the pseudo-functors $\text{Alg}_{\mathfrak{p}\mathfrak{d}}$ and $\text{Ter}_{\mathfrak{p}\mathfrak{d}}$, which sends a pair of signatures (Σ, Λ) to the category $\mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Lambda)$, and a pair of signature morphisms $(\mathfrak{d}, \mathfrak{e})$ from (Σ, Λ) to (Σ', Λ') in $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}^{\text{op}} \times \mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to the functor $\mathfrak{d}_{\mathfrak{p}\mathfrak{d}}^* \times \mathfrak{d}_{\mathfrak{p}\mathfrak{d}}^{\mathfrak{p}\mathfrak{d}}$ from $\mathbf{Alg}(\Sigma) \times \mathbf{Ter}(\Lambda)$ to $\mathbf{Alg}(\Sigma') \times \mathbf{Ter}(\Lambda')$.*

Furthermore, the family of functors $\text{Tr} = (\text{Tr}^\Sigma)_{\Sigma \in \mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}}$, together with the family $\theta = (\theta^{\mathfrak{d}})_{\mathfrak{d} \in \text{Mor}(\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}})}$, with $\theta_{\Lambda, X}^{\mathfrak{d}} = \theta_{X, A}^{\dagger\mathfrak{h}}$, is a pseudo-extranatural transformation from the pseudo-functor $\text{Alg}_{\mathfrak{p}\mathfrak{d}}(\cdot) \times \text{Ter}_{\mathfrak{p}\mathfrak{d}}(\cdot)$ to the functor $\text{K}_{\mathbf{Set}}$ from $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}^{\text{op}} \times \mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to \mathbf{Cat} that is constantly \mathbf{Set} .

Proof. We restrict ourselves to prove that, for every transformation ξ in $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ from the polyderivator \mathfrak{d} to the polyderivator \mathfrak{e} , both from Σ to Λ , we have that the following equation holds

$$\theta^{\mathfrak{e}} \circ (\text{Tr}^\Sigma * (\text{Alg}_{\mathfrak{p}\mathfrak{d}}(\xi) \times 1)) = (\text{Tr}^\Lambda * (1 \times \text{Ter}_{\mathfrak{p}\mathfrak{d}}(\xi))) \circ \theta^{\mathfrak{d}}.$$

Let $f: \mathbf{A} \longrightarrow \mathbf{B}$ be a morphism in $\mathbf{Alg}(\Lambda)$ and $P: X \longrightarrow Y$ a morphism in $\mathbf{Ter}(\Sigma)$. Then we have the following configuration

$$\begin{array}{c}
(\mathbf{A}, X) \xrightarrow{(f, P)} (\mathbf{B}, Y) \\
\swarrow \quad \searrow \\
\begin{array}{ccc}
(\mathbf{A}, \amalg_{\varphi}^{\dagger} X) \xrightarrow{(\mathbf{A}, \xi_X)} (\mathbf{A}, \amalg_{\psi}^{\dagger} X) & & (\mathbf{d}_{\text{pd}}^*(\mathbf{A}), X) \xrightarrow{(\xi^{\mathbf{A}}, X)} (\mathbf{e}_{\text{pd}}^*(\mathbf{A}), X) \\
\downarrow (f, \mathbf{d}_{\diamond}^{\text{pd}}(P)) & \downarrow (f, \mathbf{e}_{\diamond}^{\text{pd}}(P)) & \downarrow (f_{\varphi}, P) \quad \downarrow (f_{\psi}, P) \\
(\mathbf{B}, \amalg_{\varphi}^{\dagger} Y) \xrightarrow{(\mathbf{B}, \xi_Y)} (\mathbf{B}, \amalg_{\psi}^{\dagger} Y) & & (\mathbf{d}_{\text{pd}}^*(\mathbf{B}), Y) \xrightarrow{(\xi^{\mathbf{B}}, Y)} (\mathbf{e}_{\text{pd}}^*(\mathbf{B}), Y)
\end{array} \\
\swarrow \quad \searrow \\
\begin{array}{ccccc}
& (A_{\varphi})_X & \xrightarrow{(\xi^{\mathbf{A}})_X} & (A_{\psi})_X & \\
& \downarrow (\theta_{\varphi}^{\dagger})_{X,A} & & \downarrow (\theta_{\psi}^{\dagger})_{X,A} & \\
A_{\amalg_{\varphi}^{\dagger} X} & \xrightarrow{(\xi_X)^{\mathbf{A}}} & A_{\amalg_{\psi}^{\dagger} X} & & P^{\mathbf{d}_{\text{pd}}^*(\mathbf{A})} \\
& \downarrow \mathbf{d}_{\diamond}^{\text{pd}}(P)^{\mathbf{A}} & \downarrow \mathbf{e}_{\diamond}^{\text{pd}}(P)^{\mathbf{A}} & & \downarrow P^{\mathbf{e}_{\text{pd}}^*(\mathbf{A})} \\
& (A_{\varphi})_Y & \xrightarrow{(\xi^{\mathbf{A}})_Y} & (A_{\psi})_Y & \\
& \downarrow (\theta_{\varphi}^{\dagger})_{Y,A} & & \downarrow (\theta_{\psi}^{\dagger})_{Y,A} & \\
A_{\amalg_{\varphi}^{\dagger} Y} & \xrightarrow{(\xi_Y)^{\mathbf{A}}} & A_{\amalg_{\psi}^{\dagger} Y} & & (f_{\psi})_Y \\
& \downarrow f_{\amalg_{\varphi}^{\dagger} Y} & \downarrow f_{\amalg_{\psi}^{\dagger} Y} & & \downarrow (f_{\psi})_Y \\
& (B_{\varphi})_Y & \xrightarrow{(\xi^{\mathbf{B}})_Y} & (B_{\psi})_Y & \\
& \downarrow (\theta_{\varphi}^{\dagger})_{Y,B} & & \downarrow (\theta_{\psi}^{\dagger})_{Y,B} & \\
B_{\amalg_{\varphi}^{\dagger} Y} & \xrightarrow{(\xi_Y)^{\mathbf{B}}} & B_{\amalg_{\psi}^{\dagger} Y} & &
\end{array}
\end{array}$$

In the cube, the top, middle and bottom faces commute by the preceding lemma. The lateral faces commute by Lemma 8. The front face of the upper cube commutes by Proposition 69 and the front face of the lower cube commutes because f is a homomorphism. The back face of the top cube commutes because $\xi^{\mathbf{A}}$ is a homomorphism by Proposition 65, and the back face of the lower cube commutes by Proposition 66. \square

From this proposition it follows immediately the following

Corollary 19. *The quadruple $\mathfrak{Tm}_{\text{pd}} = (\mathbf{Sig}_{\text{pd}}, \text{Alg}_{\text{pd}}, \text{Ter}_{\text{pd}}, (\text{Tr}, \theta))$ is a 2-institution on the category \mathbf{Set} , the many-sorted term 2-institution of Fujiwara, or, simply, the term 2-institution of Fujiwara.*

To round off the work we have made up to this point we state in the following corollary the existence of an embedding from the 2-category of signatures, polyderivors, and transformations between polyderivors, into the 2-category of \mathcal{V} -monads, alg-morphisms, and alg-transformations between alg-morphisms.

Corollary 20. *There exists an embedding from the 2-category \mathbf{Sig}_{pd} into the 2-category $\mathbf{Mnd}_{\mathcal{V}, \text{alg}}$.*

Proof. The embedding sends

- (1) A signature Σ to the monad $(\mathbf{Set}^S, \mathbb{T}_\Sigma)$, where, we recall, $\mathbb{T}_\Sigma = (\mathbb{T}_\Sigma, \eta, \mu)$ is the standard monad derived from the adjunction $\mathbf{T}_\Sigma \dashv \mathbf{G}_\Sigma$ between the category $\mathbf{Alg}(\Sigma)$ and the category \mathbf{Set}^S , with $\mathbb{T}_\Sigma = \mathbf{G}_\Sigma \circ \mathbf{T}_\Sigma$,
- (2) A polyderivator \mathbf{d} from Σ to Λ to the alg-morphism

$$\begin{array}{ccc} \mathbf{Set}^S & \xrightarrow{\mathbb{T}_\Sigma} & \mathbf{Set}^S \\ \Downarrow \Pi_\varphi^\dagger \dashv \Delta_\varphi^\natural & \lambda & \Downarrow \Pi_\varphi^\dagger \dashv \Delta_\varphi^\natural \\ \mathbf{Set}^T & \xrightarrow{\mathbb{T}_\Lambda} & \mathbf{Set}^T \end{array}$$

also denoted by $\mathbb{T}_\mathbf{d} = (\prod_\varphi^\dagger \dashv \Delta_\varphi^\natural, \lambda)$, from $(\mathbf{Set}^S, \mathbb{T}_\Sigma)$ to $(\mathbf{Set}^T, \mathbb{T}_\Lambda)$, where the component λ_1 of the matrix λ at X is the underlying ms-mapping of $(\theta_\varphi^\natural(\eta_{\prod_\varphi^\dagger X}))^\sharp$, the canonical extension to $\mathbf{T}_\Sigma(X)$ of the ms-mapping $\Delta_\varphi^\natural(\eta_{\prod_\varphi^\dagger X}) \circ (\eta_\varphi^\natural)_X$ from X to $\Delta_\varphi^\natural(\mathbf{T}_\Lambda(\prod_\varphi^\dagger X))$, as stated in Proposition 59, and

- (3) A transformation ξ from \mathbf{d} to \mathbf{d}' to the alg-transformation

$$\begin{array}{ccc} \mathbf{Set}^S & \xrightarrow{1} & \mathbf{Set}^S \\ \Downarrow \Pi_\varphi^\dagger \dashv \Delta_\varphi^\natural & \xi & \Downarrow \Pi_{\varphi'}^\dagger \dashv \Delta_{\varphi'}^\natural \\ \mathbf{Set}^T & \xrightarrow{\mathbb{T}_\Lambda} & \mathbf{Set}^T \end{array}$$

also denoted by \mathbb{T}_ξ , from $\mathbb{T}_\mathbf{d}$ to $\mathbb{T}_{\mathbf{d}'}$, where the component ξ_0 of the matrix ξ at X is the ms-mapping ξ_X , as stated in Proposition 69. \square

From this embedding and taking into account the work by Street in [57], it follows that the polyderivators and transformations between polyderivators are a concrete foundation for a bidimensional many-sorted general algebra.

7. EQUIVALENCE OF THE SPECIFICATIONS OF HALL AND BÉNABOU.

In this section we define a 2-category of specifications, $\mathbf{Spf}_{\mathfrak{pd}}$, with objects the specifications, morphisms from a specification into a like one the polyderivators between the underlying signatures of the specifications that are compatible with the equations, and 2-cells from a morphism into a like one a convenient class of transformations between the polyderivators. In such a 2-category we prove, for every set of sorts S , the equivalence of the specifications of Hall and Bénabou for S , from which, through the pseudo-functor $\mathbf{Alg}_{\mathfrak{pd}}$, the equivalence between the corresponding categories of algebras, $\mathbf{Alg}(\mathcal{H}_S)$ and $\mathbf{Alg}(\mathcal{B}_S)$, is obtained as an easy corollary.

For a polyderivator $\mathbf{d}: \Sigma \longrightarrow \Lambda$, the functor $\mathbf{d}_\diamond^{\mathfrak{pd}}$ of translation from $\mathbf{Ter}(\Sigma)$ to $\mathbf{Ter}(\Lambda)$ enables us to define the concept of \mathfrak{pd} -specification morphism from a specification into a like one.

Definition 37. Let (Σ, \mathcal{E}) and (Λ, \mathcal{H}) be specifications. An \mathfrak{pd} -specification morphism from (Σ, \mathcal{E}) to (Λ, \mathcal{H}) is a polyderivator $\mathbf{d}: \Sigma \longrightarrow \Lambda$ such that $(\mathbf{d}_\diamond^{\mathfrak{pd}})^2[\mathcal{E}] \subseteq \mathbf{Cn}_\Lambda(\mathcal{H})$. We denote by $\mathbf{Spf}_{\mathfrak{pd}}$ the corresponding category.

Given two $\mathfrak{p}\mathfrak{d}$ -specification morphisms \mathbf{d} and \mathbf{e} from (Σ, \mathcal{E}) to (Λ, \mathcal{H}) , since \mathbf{d} and \mathbf{e} are, in particular, polyderivors from Σ to Λ , we have, in principle, at our disposal all the transformations $\xi: \mathbf{d} \rightsquigarrow \mathbf{e}$ from \mathbf{d} to \mathbf{e} as potential candidates for a concept of transformation between these $\mathfrak{p}\mathfrak{d}$ -specification morphisms.

However, the condition of commutativity for the transformations between polyderivors is too much strict, because it requires, for every formal operation $\sigma: w \longrightarrow s$ in $\Sigma_{w,s}$, the strict equality

$$\xi_s \circ d(\sigma) = e(\sigma) \circ \xi_w,$$

and, actually, what could happen (and probably the most one reasonably can hope for), as was pointed out by Fujiwara in [22], is that, under the presence of equations, such a type of equation holds only modulus the congruence generated by the equations in the target specification. Therefore, for the $\mathfrak{p}\mathfrak{d}$ -specification morphisms, the notion of transformation that we adopt, following the example of Fujiwara in [22], is that one where the strict equality between terms is replaced by the equality between them but relative to the congruence generated by the equations in the target specification. These transformations, in its turn, allow us to endow the category $\mathbf{Spf}_{\mathfrak{p}\mathfrak{d}}$ of a structure of 2-category.

Definition 38. Let \mathbf{d} and $\mathbf{e}: (\Sigma, \mathcal{E}) \longrightarrow (\Lambda, \mathcal{H})$ be $\mathfrak{p}\mathfrak{d}$ -specification morphisms. A *transformation from \mathbf{d} to \mathbf{e}* is a choice function ξ for $(\mathbf{B}\mathrm{Ter}_T(\Lambda)_{\varphi(s), \psi(s)})_{s \in S}$, such that, for every formal operation $\sigma: w \longrightarrow s$, we have that

$$\xi_s \circ d(\sigma) \equiv_{\overline{\mathcal{H}}} e(\sigma) \circ \xi_w.$$

Proposition 72. *The specifications, the $\mathfrak{p}\mathfrak{d}$ -specification morphisms, and the transformations between $\mathfrak{p}\mathfrak{d}$ -specification morphisms determine a 2-category $\mathbf{Spf}_{\mathfrak{p}\mathfrak{d}}$.*

The pseudo-functor $\mathrm{Alg}_{\mathfrak{p}\mathfrak{d}}$ from $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to \mathbf{Cat} can be lifted up to the 2-category $\mathbf{Spf}_{\mathfrak{p}\mathfrak{d}}$, by assigning, in particular, to a specification (Σ, \mathcal{E}) the category $\mathbf{Alg}(\Sigma, \mathcal{E})$ of its models.

Proposition 73. *There exists a pseudo-functor $\mathrm{Alg}_{\mathfrak{p}\mathfrak{d}}^{\mathrm{SP}}$ from $\mathbf{Spf}_{\mathfrak{p}\mathfrak{d}}$ to \mathbf{Cat} defined as follows*

- (1) $\mathrm{Alg}_{\mathfrak{p}\mathfrak{d}}^{\mathrm{SP}}$ sends a specification (Σ, \mathcal{E}) to the category $\mathrm{Alg}_{\mathfrak{p}\mathfrak{d}}^{\mathrm{SP}}(\Sigma, \mathcal{E}) = \mathbf{Alg}(\Sigma, \mathcal{E})$ of its models, i.e., the full subcategory of $\mathbf{Alg}(\Sigma)$ determined by those Σ -algebras that satisfy all the equations in \mathcal{E} .
- (2) $\mathrm{Alg}_{\mathfrak{p}\mathfrak{d}}^{\mathrm{SP}}$ sends a $\mathfrak{p}\mathfrak{d}$ -specification morphism \mathbf{d} from (Σ, \mathcal{E}) to (Λ, \mathcal{H}) to the functor $\mathrm{Alg}_{\mathfrak{p}\mathfrak{d}}^{\mathrm{SP}}(\mathbf{d}) = \mathbf{d}_{\mathfrak{p}\mathfrak{d}}^*$ from $\mathbf{Alg}(\Lambda, \mathcal{H})$ to $\mathbf{Alg}(\Sigma, \mathcal{E})$, obtained from the functor $\mathbf{d}_{\mathfrak{p}\mathfrak{d}}^*$ from $\mathbf{Alg}(\Lambda)$ to $\mathbf{Alg}(\Sigma)$ by bi-restriction.
- (3) $\mathrm{Alg}_{\mathfrak{p}\mathfrak{d}}^{\mathrm{SP}}$ sends a transformation $\xi: \mathbf{d} \rightsquigarrow \mathbf{e}$ from \mathbf{d} to \mathbf{e} to the natural transformation $\mathrm{Alg}_{\mathfrak{p}\mathfrak{d}}(\xi)$ from $\mathbf{d}_{\mathfrak{p}\mathfrak{d}}^*$ to $\mathbf{e}_{\mathfrak{p}\mathfrak{d}}^*$.

The pseudo-functor $\mathrm{Ter}_{\mathfrak{p}\mathfrak{d}}$ from $\mathbf{Sig}_{\mathfrak{p}\mathfrak{d}}$ to \mathbf{Cat} can also be lifted up to the 2-category $\mathbf{Spf}_{\mathfrak{p}\mathfrak{d}}$.

Proposition 74. *There exists a pseudo-functor $\mathrm{Ter}_{\mathfrak{p}\mathfrak{d}}^{\mathrm{SP}}$ from $\mathbf{Spf}_{\mathfrak{p}\mathfrak{d}}$ to \mathbf{Cat} defined as follows*

- (1) $\mathrm{Ter}_{\mathfrak{p}\mathfrak{d}}^{\mathrm{SP}}$ sends a specification (Σ, \mathcal{E}) to the category $\mathrm{Ter}_{\mathfrak{p}\mathfrak{d}}^{\mathrm{SP}}(\Sigma, \mathcal{E}) = \mathbf{Ter}(\Sigma, \mathcal{E})$, where $\mathbf{Ter}(\Sigma, \mathcal{E})$ is the quotient category $\mathbf{Ter}(\Sigma)/\overline{\mathcal{E}}$.
- (2) $\mathrm{Ter}_{\mathfrak{p}\mathfrak{d}}^{\mathrm{SP}}$ sends a $\mathfrak{p}\mathfrak{d}$ -specification morphism \mathbf{d} from (Σ, \mathcal{E}) to (Λ, \mathcal{H}) to the functor $\mathrm{Ter}_{\mathfrak{p}\mathfrak{d}}^{\mathrm{SP}}(\mathbf{d})$ from the quotient category $\mathbf{Ter}(\Sigma, \mathcal{E}) = \mathbf{Ter}(\Sigma)/\overline{\mathcal{E}}$ to the quotient category $\mathbf{Ter}(\Lambda, \mathcal{H}) = \mathbf{Ter}(\Lambda)/\overline{\mathcal{H}}$, which assigns to a morphism $[P]_{\overline{\mathcal{E}}}$ from X to Y in $\mathbf{Ter}(\Sigma, \mathcal{E})$ the morphism

$$\mathrm{Ter}_{\mathfrak{p}\mathfrak{d}}^{\mathrm{SP}}(\mathbf{d})([P]_{\overline{\mathcal{E}}}) = [\mathbf{d}_{\mathfrak{p}\mathfrak{d}}^{\mathfrak{p}\mathfrak{d}}(P)]_{\overline{\mathcal{H}}}: \coprod_{\varphi}^{\dagger} X \longrightarrow \coprod_{\varphi}^{\dagger} Y$$

in $\mathbf{Ter}(\mathbf{A}, \mathcal{H})$.

- (3) $\mathbf{Ter}_{\mathbf{pd}}^{\text{SP}}$ sends a transformation $\xi: \mathbf{d} \rightsquigarrow \mathbf{e}$ from \mathbf{d} to \mathbf{e} to the natural transformation $\mathbf{Ter}_{\mathbf{pd}}(\xi)$ from $\mathbf{Ter}_{\mathbf{pd}}^{\text{SP}}(\mathbf{d})$ to $\mathbf{Ter}_{\mathbf{pd}}^{\text{SP}}(\mathbf{e})$.

Furthermore, from the 2-category $\mathbf{Spf}_{\mathbf{pd}}^{\text{OP}} \times \mathbf{Spf}_{\mathbf{pd}}$ to the 2-category \mathbf{Cat} there exists a pseudo-functor $\mathbf{Alg}_{\mathbf{pd}}^{\text{SP}}(\cdot) \times \mathbf{Ter}_{\mathbf{pd}}^{\text{SP}}(\cdot)$ and a pseudo-extranatural transformation $(\text{Tr}^{\text{SP}}, \theta^{\text{SP}})$ from $\mathbf{Alg}_{\mathbf{pd}}^{\text{SP}}(\cdot) \times \mathbf{Ter}_{\mathbf{pd}}^{\text{SP}}(\cdot)$ to the functor constantly \mathbf{Set} , and from this we get the following

Corollary 21. *The quadruple $\mathfrak{Spf}_{\mathbf{pd}} = (\mathbf{Spf}_{\mathbf{pd}}, \mathbf{Alg}_{\mathbf{pd}}^{\text{SP}}, \mathbf{Ter}_{\mathbf{pd}}^{\text{SP}}, (\text{Tr}^{\text{SP}}, \theta^{\text{SP}}))$ is a 2-institution on the category \mathbf{Set} , the so-called many-sorted specification 2-institution of Fujiwara, or, simply, the specification 2-institution of Fujiwara.*

From the pseudo-functor $\mathbf{Alg}_{\mathbf{pd}}^{\text{SP}}$, from $\mathbf{Spf}_{\mathbf{pd}}$ to \mathbf{Cat} , to the pseudo-functor $\mathbf{Alg}_{\mathbf{pd}} \circ \text{sig}^{\text{OP}}$, between the same 2-categories, there exists a pseudo-natural transformation, $\text{In}_{\mathbf{pd}}$, which sends a specification (Σ, \mathcal{E}) to the full embedding $\text{In}_{(\Sigma, \mathcal{E})}$ of $\mathbf{Alg}(\Sigma, \mathcal{E})$ into $\mathbf{Alg}(\Sigma)$. Besides, from the pseudo-functor $\mathbf{Ter}_{\mathbf{pd}} \circ \text{sig}$, from $\mathbf{Spf}_{\mathbf{pd}}$ to \mathbf{Cat} , to the pseudo-functor $\mathbf{Ter}_{\mathbf{pd}}^{\text{SP}}$, between the same 2-categories, there exists a (strict) pseudo-natural transformation, $\text{Pr}_{\mathbf{pd}}$, given by the following data

- (1) For each specification (Σ, \mathcal{E}) , the projection functor $\text{Pr}_{\bar{\mathcal{E}}}$ from $\mathbf{Ter}(\Sigma)$ to the quotient category $\mathbf{Ter}(\Sigma)/\bar{\mathcal{E}}$.
- (2) For each specification morphism \mathbf{d} from (Σ, \mathcal{E}) to $(\mathbf{A}, \mathcal{H})$, the isomorphic identity natural transformation, denoted in this case by $\text{Pr}_{\mathbf{d}}$, from the functor $\text{Pr}_{\bar{\mathcal{H}}} \circ (\text{Ter} \circ \text{sig})(\mathbf{d})$, from the category $\mathbf{Ter}(\Sigma)$ to the category $\mathbf{Ter}(\mathbf{A})/\bar{\mathcal{H}}$, to the functor $\mathbf{Ter}^{\text{SP}}(\mathbf{d}) \circ \text{Pr}_{\bar{\mathcal{E}}}$, between the same categories.

Therefore we have obtained the following

Corollary 22. *The pair $(\text{sig}, (\text{In}_{\mathbf{pd}}, \text{Pr}_{\mathbf{pd}}))$ is a morphism of 2-institutions from the 2-institution $\mathfrak{Spf}_{\mathbf{pd}}$ to the 2-institution $\mathfrak{Im}_{\mathbf{pd}}$.*

Our next goal is to prove that the specifications of Bénabou and Hall are equivalent in the 2-category $\mathbf{Spf}_{\mathbf{pd}}$.

Proposition 75. *The specifications $\text{B}_S = (\Sigma^{\text{B}_S}, \mathcal{E}^{\text{B}_S})$, of Bénabou for S , and $\text{H}_S = (\Sigma^{\text{H}_S}, \mathcal{E}^{\text{H}_S})$, of Hall for S , are equivalent in the 2-category $\mathbf{Spf}_{\mathbf{pd}}$.*

Proof. Let S be a set of sorts. From the signature Σ^{B_S} to the signature Σ^{H_S} , we have the polyderivator $\mathbf{d} = (\varphi, d)$, where φ is the mapping

$$\begin{aligned} S^* \times S^* &\longrightarrow (S^* \times S)^* \\ (u, v) &\longmapsto ((u, v_0), \dots, (u, v_{|v|-1})) \end{aligned}$$

while $d: \Sigma^{\text{B}_S} \longrightarrow \text{BTer}_{S^* \times S}(\Sigma^{\text{H}_S})_{\varphi^\sharp \times \varphi}$ is defined as

- (1) For every $w \in S^*$, and $i \in |w|$,

$$d(\pi_i^w) = (\pi_i^w).$$

- (2) For every $u, w \in S^*$,

$$d(\langle \rangle_{u, w}) = (v_0^{u, w_0}, \dots, v_{|w|-1}^{u, w_{|w|-1}}).$$

- (3) For every $u, v, w \in S^*$,

$$\begin{aligned} d(\circ_{u, v, w}) &= (\xi_{u, v, w_0}(v_{|v|}^{u, w_0}, v_0^{u, v_0}, \dots, v_{|v|-1}^{u, v_{|v|-1}}), \dots, \\ &\quad \xi_{u, v, w_{|w|-1}}(v_{|v|+|w|-1}^{u, w_{|w|-1}}, v_0^{u, v_0}, \dots, v_{|v|-1}^{u, v_{|v|-1}})). \end{aligned}$$

Now we prove that the definition of \mathbf{d} is sound. For the operations $\pi_i^w \in \Sigma_{\lambda, (w, (w_i))}^{\mathbf{B}S}$, we have that

$$\begin{aligned} d(\pi_i^w) &\in \text{BTer}_{S^* \times S}(\Sigma^{\mathbf{H}S})_{\varphi^\#(\lambda), \varphi(w, (w_i))} \\ &= \text{BTer}_{S^* \times S}(\Sigma^{\mathbf{H}S})_{\lambda, ((w, (w_i)))} \\ &= \text{T}_{\Sigma^{\mathbf{H}S}}(\downarrow \lambda)_{((w, (w_i)))}, \end{aligned}$$

because $d(\pi_i^w)$ is a word of length 1 that has as its unique component an operation of coarity $(w, (w_i))$.

For the operations $\langle \rangle_{u, w} \in \Sigma_{((u, (w_0)), \dots, (u, (w_{|w|-1}))), (u, w)}^{\mathbf{B}S}$, we have that

$$\begin{aligned} d(\langle \rangle_{u, w}) &\in \text{BTer}_{S^* \times S}(\Sigma^{\mathbf{H}S})_{\varphi^\#((u, w_0), \dots, (u, (w_{|w|-1}))), \varphi(u, w)} \\ &= \text{BTer}_{S^* \times S}(\Sigma^{\mathbf{H}S})_{((u, w_0), \dots, (u, w_{|w|-1})), ((u, w_0), \dots, (u, w_{|w|-1}))} \\ &= \text{T}_{\Sigma^{\mathbf{H}S}}(\downarrow((u, w_0), \dots, (u, w_{|w|-1})))_{((u, w_0), \dots, (u, w_{|w|-1}))}, \end{aligned}$$

because $d(\langle \rangle_{u, w})$ is a word of length $|w|$ that, for every $i \in |w|$, has as i -th component a term of coarity $(u, (w_i))$. For the operations $\circ_{u, v, w} \in \Sigma_{((u, v), (v, w)), (u, w)}^{\mathbf{B}S}$, we have that

$$\begin{aligned} d(\circ_{u, v, w}) &\in \text{BTer}_{S^* \times S}(\Sigma^{\mathbf{H}S})_{\varphi^\#((u, v), (v, w)), \varphi(u, w)} \\ &= \text{BTer}_{S^* \times S}(\Sigma^{\mathbf{H}S})_{((u, v_0), \dots, (u, v_{|v|-1}), (v, w_0), \dots, (v, w_{|w|-1})), ((u, w_i) | i \in |w|)} \\ &= \text{T}_{\Sigma^{\mathbf{H}S}}(\downarrow(((u, v_j) | j \in |v|), ((v, w_i) | i \in |w|)))_{((u, w_i) | i \in |w|)}, \end{aligned}$$

because $d(\circ_{u, v, w})$ is a word of length $|w|$ that, for every $i \in |w|$, has as i -th component a term of coarity (u, w_i) .

From the signature $\Sigma^{\mathbf{H}S}$ to the signature $\Sigma^{\mathbf{B}S}$ we have the polyderivator $\mathbf{e} = (\psi, e)$, where ψ is the mapping

$$\begin{aligned} S^* \times S &\longrightarrow (S^* \times S)^* \\ (w, s) &\longmapsto ((w, (s))) \end{aligned}$$

while $e: \Sigma^{\mathbf{H}S} \longrightarrow \text{BTer}_{S^* \times S}(\Sigma^{\mathbf{B}S})_{\psi^\# \times \psi}$ is defined as

(1) For every $w \in S^*$ and $i \in |w|$,

$$e(\pi_i^w) = (\pi_i^w).$$

(2) For every $u, w \in S^*$ and $s \in S$,

$$e(\xi_{u, w, s}) = (v_0^{w, s} \circ \langle v_1^{u, w_0}, \dots, v_{|w|}^{u, w_{|w|-1}} \rangle).$$

The polyderivators \mathbf{d} and \mathbf{e} are, obviously, compatible with the respective equations, hence are \mathbf{pd} -specification morphisms.

Finally we should prove that there are invertible transformations between the identity at $\Sigma^{\mathbf{B}S}$ and the polyderivator $\mathbf{e} \circ \mathbf{d}$, as well as between the identity at $\Sigma^{\mathbf{H}S}$ and the polyderivator $\mathbf{d} \circ \mathbf{e}$. Since both proofs are analogous, we restrict ourselves to present only the first one.

From the identity at $\Sigma^{\mathbf{B}S}$ into $\mathbf{e} \circ \mathbf{d}$ we have the transformation χ , defined, for every $(u, w) \in S^* \times S^*$, as the term

$$\chi_{(u, w)} = \langle \pi_0^w \circ v_0, \dots, \pi_{|w|-1}^w \circ v_0 \rangle \in \text{T}_{\Sigma^{\mathbf{B}S}}(\downarrow((u, w)))_{((u, (w_0)), \dots, (u, (w_{|w|-1})))},$$

and from $\mathbf{e} \circ \mathbf{d}$ into the identity at $\Sigma^{\mathbf{B}S}$ we have the transformation ρ , defined, for every $(u, w) \in S^* \times S^*$, as the term

$$\rho_{(u, w)} = \langle v_0, \dots, v_{|w|-1} \rangle_{u, w} \in \text{T}_{\Sigma^{\mathbf{B}S}}(\downarrow((u, (w_0)), \dots, (u, (w_{|w|-1}))))_{((u, w))}.$$

Then $\rho_{(u, w)} \circ \chi_{(u, w)}$ is the term

$$\langle \pi_0^w \circ v_0, \dots, \pi_{|w|-1}^w \circ v_0 \rangle_{u, w} = v_0,$$

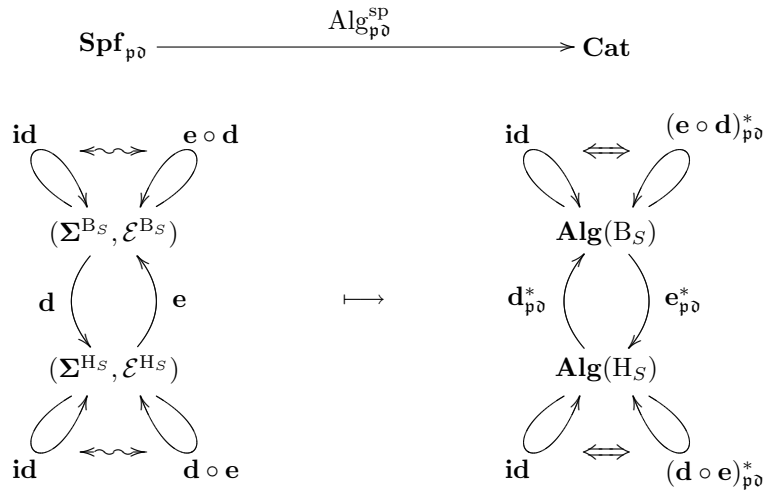
and $\chi_{(u,w)} \circ \rho_{(u,w)}$ is the term

$$(\pi_0^w \circ \langle v_0, \dots, v_{|w|-1} \rangle_{u,w}, \dots, \pi_{|w|-1}^w \circ \langle v_0, \dots, v_{|w|-1} \rangle_{u,w}) = (v_0, \dots, v_{|w|-1}),$$

hence ξ and ρ are inverses. □

Corollary 23. *For every set of sorts S , the category $\mathbf{Alg}(\mathbf{H}_S)$, of Hall algebras for S , is equivalent to the category $\mathbf{Alg}(\mathbf{B}_S)$, of Bénabou algebras for S .*

Proof. It follows from the existence of the pseudo-functor $\mathbf{Alg}_{\mathbf{pd}}^{\mathbf{sp}}$ from the 2-category $\mathbf{Spf}_{\mathbf{pd}}$ to the 2-category \mathbf{Cat} and from the fact that the specifications $(\Sigma^{\mathbf{B}_S}, \mathcal{E}^{\mathbf{B}_S})$ and $(\Sigma^{\mathbf{H}_S}, \mathcal{E}^{\mathbf{H}_S})$ are equivalent in the 2-category $\mathbf{Spf}_{\mathbf{pd}}$. We summarize these facts by means of the following picture:



□

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UNIVERSIDAD DE VALENCIA, DEPARTAMENTO DE LÓGICA Y FILOSOFÍA DE LA CIENCIA, E-46010
VALENCIA, SPAIN
E-mail address: Juan.B.Climent@uv.es

UNIVERSIDAD DE VALENCIA, DEPARTAMENTO DE LÓGICA Y FILOSOFÍA DE LA CIENCIA, E-46010
VALENCIA, SPAIN
E-mail address: Juan.Soliveres@uv.es